

Generalized Kähler Taub-NUTs and Two Exceptional Instantons

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Abstract

We study the one-parameter family of twisted Kähler Taub-NUT metrics (discovered by Donaldson), along with two exceptional Taub-NUT-like instantons, and understand them to the extent that should be sufficient for blow-up and gluing arguments. In particular we parametrize their geodesics from the origin, determine curvature fall-off rates and volume growth rates, find blow-down limits, and show that none of these instantons is ALF, except the standard Taub-NUT itself.

Contents

1	Introduction	2
1.1	Description of the Kähler Reduction	3
1.1.1	The moment description and classification	3
1.1.2	The exceptional half-plane instanton	3
1.1.3	The Taub-NUT metrics	4
1.2	Description of Results	4
2	Overview of Kähler reduction	7
2.1	Polytope construction	7
2.2	Metric quantities	8
2.3	Curvature quantities	8
2.4	The classification	9
2.4.1	The polytope outline	9
2.4.2	Generalized Taub-NUTs and the exceptional Taub-NUT	10
2.4.3	Half-plane metrics	11
2.4.4	Planar metrics	11
3	The generalized Taub-NUT metrics	11
3.1	Coordinates	11
3.2	Distance functions and geodesic normal coordinates	12
3.2.1	The distance functions S_η	12
3.2.2	The geodesics based at the origin	13
3.2.3	Geodesic Normal Coordinates	14

3.2.4	Asymptotic approximations of functions	15
3.3	Computation of the key asymptotic ratios	18
3.3.1	Asymptotic Volume Growth	19
3.3.2	Asymptotic Sectional Curvature, and energy computation	20
3.4	Three kinds of Blowdown	22
3.4.1	The Gromov-Hausdorff Blowdown	22
3.4.2	Two generalized blowdowns	23
3.4.3	Metric and coordinate convergence under blowdown	24
3.4.4	Distance functions on the generalized blow-downs	26
4	The exceptional Taub-NUT	26
4.1	Coordinates	26
4.2	Distance functions and geodesic normal coordinates	27
4.3	Volume and curvature computations	29
4.4	A scaled and an unscaled pointed limit	30
4.4.1	The blowdown	30
4.4.2	An unscaled pointed limit	30
5	The exceptional half-plane instanton	31

1 Introduction

In this note we collect sufficiently detailed information on the Kähler Taub-NUT class of metrics, discovered by Donaldson [5] and further studied by Abreu and Sena-Dias [3] [14], to meet the needs of gluing constructions and blow-up arguments. We examine asymptotics such as curvature fall-off and asymptotic volume growth, and we compute L^2 curvature energy in a fairly simple way. We show that both exceptional instantons have infinite L^2 energy and that, with the exception of the standard Taub-NUT, none of these instantons is actually ALF.

The approach in this paper is through the theory of toric Kähler geometry. The instantons $(M^4, J, \omega, \mathcal{X}_1, \mathcal{X}_2)$ we consider are scalar-flat toric Kähler 4-manifolds where $\mathcal{X}_1, \mathcal{X}_2$ are commuting real-holomorphic Killing fields, and the complex manifold (M^4, J) is $\mathbb{C} \times \mathbb{C}$ with one of three obvious symmetry structures: translation on both factors, translation on one factor and rotation on the other, and rotation on both factors.

In [16], complete, scalar flat manifolds with commuting holomorphic Killing fields were classified, and it was found that the metrics written down in [3] were in fact all such metrics. It was found that when the symmetries have either one or no common fixed points, the complex manifold must be $\mathbb{C} \times \mathbb{C}$. Its metric, up to homothety, must be one of the following: in the case of two translational symmetries it is the flat metric; in the case of one translational and one rotational symmetry, it is the flat metric or a certain exceptional instanton that we call the exceptional half-plane instanton; and in the case of two rotational symmetries there are three possibilities: it may be flat, it may be in the one-parameter family of generalized Taub-NUT instantons, or it may be the exceptional Taub-NUT instanton.

1.1 Description of the Kähler Reduction

In this section we sketch the basics of the objects under study: the generalized Taub-NUTs, the exceptional Taub-NUT, and the exceptional half-plane instanton.

1.1.1 The moment description and classification

By a construction originating in classical mechanics, the existence of symplectomorphisms $\mathcal{X}_1, \mathcal{X}_2$ allows construction of “action-angle” coordinates $(\varphi^1, \varphi^2, \theta^1, \theta^2)$, where the “angles” θ^1, θ^2 parametrize the integrated flows of $\mathcal{X}_1, \mathcal{X}_2$, and the “actions” φ^1, φ^2 parametrize \mathcal{X}_1 - \mathcal{X}_2 leaves themselves.

Any metric g_4 on one of our 4-dimensional instantons can be written, explicitly though unenlighteningly, in these coordinates via methods of [3] or [16], as explained in Section 2. If one “forgets” the angle directions and retains only the action variables φ^1, φ^2 , one obtains the *moment map* $M^4 \rightarrow \Sigma^2$, whose image is naturally a polygon in the φ^1 - φ^2 plane with an inherited metric; see Section 2. The resulting object (Σ^2, g_Σ) is called the metric polytope.

In [5],[3],[16], instantons are studied, in part, by using a set of coordinates called *volumetric normal coordinates*. If we denote these by (x, y) or (x^1, x^2) , we define

$$x = x^1 = \sqrt{|\mathcal{X}_1|^2 |\mathcal{X}_2|^2 - \langle \mathcal{X}_1, \mathcal{X}_2 \rangle^2}, \quad (1)$$

which is the parallelogram volume of $\{\mathcal{X}_1, \mathcal{X}_2\}$, which, remarkably, is harmonic in the natural polytope metric g_Σ . Then y is just defined as the harmonic conjugate of x . The map $z : \Sigma^2 \rightarrow \mathbb{C}$ where $z = x + \sqrt{-1}y$ is analytic, and if the polytope has connected boundary, it is an unbranched map onto the right half-plane $H^2 \subset \mathbb{C}$ and the polytope boundary $\partial\Sigma^2$ maps 1-1 onto the imaginary axis. Then φ^1, φ^2 can be expressed in terms of x, y , and the metrics g_Σ, g_4 can be written down explicitly in terms of the transition. Indeed in (x, y) -coordinates, we have simply $g_{\Sigma ij} = \frac{1}{x^4} \det \left(\frac{\partial \varphi^k}{\partial x^l} \right) \delta_{ij}$.

The moment variables φ^1, φ^2 are constrained by the PDE

$$x \left(\varphi_{xx}^i + \varphi_{yy}^i \right) - \varphi_x = 0. \quad (2)$$

In [16], a Liouville-type theorem was used to classify solutions to this degenerate-elliptic system under the condition that the corresponding polytope have connected boundary.

1.1.2 The exceptional half-plane instanton

It was proven that, after possible affine recombination, the one-parameter family of solutions

$$\varphi^1 = \frac{1}{2}x^2, \quad \varphi^2 = y + \frac{M}{2}yx^2 \quad (3)$$

are the only solutions that produce the half-plane polytope; the parameter M simply scales the metric and the choice $M = 0$ is the flat instanton.

The resulting 4-dimensional instanton is called the exceptional half-plane instanton. The polytope metric is $g_\Sigma = (1 + \frac{M}{2}x^2)((dx)^2 + (dy)^2)$. The instanton metric g_4 and a description of its properties is in Section 5.

1.1.3 The Taub-NUT metrics

This is the case of the quarter-plane polytope (after possible affine recombination of φ^1, φ^2). The two-parameter family of solutions

$$\varphi^1 = \frac{1}{\sqrt{2}} \left(-y + \sqrt{x^2 + y^2} \right) + \frac{\alpha}{2} x^2, \quad \varphi^2 = \frac{1}{\sqrt{2}} \left(y + \sqrt{x^2 + y^2} \right) + \frac{\beta}{2} x^2, \quad (4)$$

are the only solutions that generate the quarter-plane polytope. These were written down by Donaldson [5]. The corresponding 4-dimensional instantons are the generalized Taub-NUTs. If we set $M = \frac{\alpha+\beta}{2}$ and $k = \frac{\alpha-\beta}{\alpha+\beta}$, the polytope metric is

$$g_\Sigma = \frac{1 + \sqrt{2}M \left(k y + \sqrt{x^2 + y^2} \right)}{\sqrt{x^2 + y^2}} (dx \otimes dx + dy \otimes dy). \quad (5)$$

Once again the parameter $M > 0$ is simply scale, as can be seen more clearly in (7). As $M \rightarrow 0$, any of these instantons converges on the flat instanton. The case $M = \sqrt{2}$ is the standard scale, where $\sup |\sec| = 1$. The parameter $k \in [-1, 1]$, called the instanton's *chirality number*, parametrizes the family of inequivalent Taub-NUT metrics themselves. The instantons given by k and $-k$ are isometric, whereas the corresponding polytope metrics on Σ^2 are enantiometric (isometric but with flipped orientation). The case $k = 0$ is the standard (achiral, Ricci-flat) Taub-NUT, and the extreme case $k = 1, -1$ is the exceptional Taub-NUT. Choices of k outside the $[-1, 1]$ range produce both topological and curvature singularities.

1.2 Description of Results

The first step in this paper is choosing better isothermal coordinates than the volumetric normal coordinates. We change to *adapted quadratic coordinates* u, v via the fourth-degree polynomial transitions

$$\varphi^1 = \frac{v^2}{M} (1 + (1+k)u^2), \quad \varphi^2 = \frac{u^2}{M} (1 + (1-k)v^2), \quad (6)$$

which is a diffeomorphism of the first quadrant to itself; see Sections 3 and 4. One can show that (u, v) is actually an isothermal system; see Section 3.1. The metric expressed in $(u, v, \theta^1, \theta^2)$ coordinates is

$$g = \frac{2\sqrt{2}}{M} \left[(1 + (1+k)u^2 + (1-k)v^2) (du^2 + dv^2) + G_{ij} d\theta^i \otimes d\theta^j \right] \quad (7)$$

where the matrix G is a function of u, v and the parameter k whose particular form is unimportant for now (we write it down in (31)). Here we see M explicitly as a scale.

We find that the Killing field $\mathcal{X} = (1-k)\mathcal{X}_1 - (1+k)\mathcal{X}_2$ is everywhere an eigenvector of $G_{ij}d\theta^i \otimes d\theta^j$. The corresponding eigenvalue approaches 1 asymptotically, so this is indeed the collapsing field at infinity, in the sense of Cheeger-Gromov collapsing theory [4]. The three remaining eigendirections of the metric are not represented by Killing fields, and their corresponding eigenvalues grow linearly with distance, asymptotically.

To find parametrized geodesics from the origin, we use the form of the metric in (7) and a separation method to solve the Eikonal equation $|\nabla S| = 1$ *explicitly* in a certain variety of cases. We find that enough such solutions to allow explicit parametrization of all geodesics based at the origin. Thus we can explicitly compute the polytope metric in exponential polar coordinates (equation (47)), and explicitly compute the key asymptotic properties.

We summarize our results in the following. The first theorem is well-known and is included for completeness.

Theorem 1.1 (The standard Taub-NUT) *These are the metrics of (7) with $k = 0$. The collapsing field at infinity is $\mathcal{X}_1 + \mathcal{X}_2$. These metrics are Ricci-flat and have total curvature*

$$\int |\text{Rm}|^2 = 32\pi^2. \quad (8)$$

Volume growth of geodesic balls is cubic: $\text{Vol } B(R) = O(R^3)$ and curvature decay is cubic: $|\text{Ric}| = O(R^{-3})$. Its Gromov-Hausdorff blowdown is flat \mathbb{R}^3 .

Theorem 1.2 (The chiral Taub-NUTs) *These are the metrics (7) with $k \in (-1, 0) \cup (0, 1)$. The collapsing field at infinity is $(1+k)\mathcal{X}_1 + (1-k)\mathcal{X}_2$. These manifolds are scalar-flat and half-conformally flat, and have total energy*

$$\int |\text{Ric}|^2 = 4\pi^2 \frac{k^2}{1-k^2}, \quad \int |\text{Rm}|^2 = 16\pi^2 \frac{2-k^2}{1-k^2}. \quad (9)$$

Volume growth of geodesic balls is precisely cubic: $\text{Vol } B(R) = O(R^3)$, and curvature decay is quadratic: $|\text{Ric}| = O(R^{-2})$, $|\text{Rm}| = O(R^{-2})$. Gromov-Hausdorff blowdowns are non-flat and have a curvature singularity point. The limit is either a 3-dimensional stratified orbifold (when $\frac{1-k}{1+k}$ is rational) or is the closed half-plane in (when $\frac{1-k}{1+k}$ is irrational).

Theorem 1.3 (The exceptional Taub-NUT) *This the metric with maximum chirality $k = 1$ or -1 . This manifold is scalar-flat and half-conformally flat, and has infinite L^2 -norm of both $|\text{Ric}|$ and $|\text{Rm}|$. Growth of geodesic balls is quartic: $\text{Vol } B(R) = O(R^4)$. Sectional curvature decays quadratically $|\text{Rm}| = O(R^{-2})$ along all geodesics from the origin, except for the family of geodesic rays that make up an exceptional, totally geodesic codimension 2 holomorphic submanifold containing the origin. Along these rays, curvature does not decay: $|\text{Rm}| = O(1)$.*

Theorem 1.4 (The exceptional half-plane instanton) *In volumetric normal coordinates this instanton has polytope metric*

$$g_\Sigma = (1+x^2)(dx \otimes dx + dy \otimes dy), 0 \leq x < \infty, -\infty < y < \infty \quad (10)$$

which is formally identical to the polytope metric of the exceptional Taub-Nut metric if the latter is expressed in adapted quadratic coordinates. This instanton is scalar flat and half-conformally flat, and has $\int |\text{Ric}|^2 = \int |\text{Rm}|^2 = \infty$. Volume growth is quartic: $\text{Vol } B(R) = O(R^4)$. Curvature decay is generically $O(R^{-2})$, but along the geodesics that lie in a totally-geodesic 2-dimensional submanifold, it has no curvature decay.

Remark. The exceptional Taub-NUT and the exceptional half-plane instanton have eerily similar properties, so one would wonder if they are the same, or perhaps if one is the cover of the other. But they are in fact completely different, as we prove at the end of Section 5. The following remark indicates there is indeed a relationship between them, of a different sort.

Remark. The exceptional Taub-NUT has rays along which curvature does not decay. The injectivity radius does not collapse along any such ray, so it is interesting to know what the pointed Gromov-Hausdorff limit is. This limit is actually the exceptional half-plane instanton. We explore this in Section 4.

Remark. All instantons considered above are simply-connected except the exceptional half-plane instanton, whose π_1 is represented by a torus meridian in any of the torus fibers. Thus we could consider this to be two examples: a simply connected half-plane instanton with cylinder fibers, and a half-plane instanton with torus fibers and $\pi_1 = \mathbb{Z}$.

Remark. Although all these instantons have just one collapsing field at infinity and cubic asymptotic volume growth, blow-down limits could be either two or three dimensional. This strange-seeming phenomenon is related to the way Berger spheres can collapse either to a 2-sphere (possibly with orbifold points) or to a line segment, by scaling just *one* direction of a left-invariant metric, depending on whether that direction has rational or irrational slope within the Hopf tori. This is explored in Section 3.4, where we also draw two other kinds of limits (not Gromov-Hausdorff limits) that correct this strangeness.

Remark. Taub-NUT metrics have long proved a source of examples in both general relativity [11] [15] [10] and Riemannian geometry, where Taub-NUT metrics have been generalized in numerous ways. There are the multi-Taub-NUT metrics of Gibbons-Hawking [6]; these are hyperkähler, and many of them are toric, but the only moment polytope with one vertex is the standard Taub-NUT. Page [13] explores Euclidean Taub-NUT metrics with a magnetic anomaly, which are Ricci-flat but not half-conformally flat, but which have curvature singularities. Noriaki-Toshihiro [12] explore classes of “generalized Taub-NUT” and “extended Taub-NUT” metrics which have torus symmetry. Like examples considered in this paper, some of their examples are half-conformally flat and non-Einstein. However none of their examples are Kähler, except the standard one.

Remark. We claim explicit solutions of the geodesic equations, but we should say what is meant by “explicit.” Our separation method lets us write down *unparametrized* geodesics with simple algebraic expressions (see equation (37)). The parametrization is given by a transcendental, not an algebraic expression; one must invert a function of the type $f(x) =$

$x + \log(x)$ (see equation (43)). However, near infinity we are able to approximate even the parametrization with a simple algebraic expression to arbitrary closeness; see Section 3.2.4) and Corollary 3.3.

Remark. All toric scalar-flat metrics on the $\mathcal{O}(-k)$ bundles over \mathbb{P}^1 are now known as well; they were written down in [3] and the results of [16] show that these are all such metrics. In [3] it was shown exactly which of these are Einstein. Of course the ALE Einstein metrics were already known [9]. It would be interesting to learn more about the non-Einstein metrics on these spaces, in particular which of them are ALF—a reasonable conjecture is that these are precisely the multi-Taub-NUTs and the multi-Eguchi-Hanson instantons (that is, precisely the hyperkähler instantons). All of the $\mathcal{O}(-k)$ metrics have identical polytopes, up to affine recombination of φ^1, φ^2 , so one would like specific criteria, in terms of properties of the metric polytope itself, for how the different $\mathcal{O}(-k)$ bundles are distinguished, and also which polytope metrics correspond to actual instantons.

2 Overview of Kähler reduction

We review the moment construction. This encodes all $(M^4, J, \omega, \mathcal{X}_1, \mathcal{X}_2)$ data into the metric of a real 2-dimensional polytope (Σ^2, g_Σ) . This expository section, included for the reader's convenience, has little new except the discussion of Ricci curvature in Section 2.3, where we introduce the Ricci pseudopotentials and the Ricci pseudo-volume form on the polytope. See [1] [7] [5] [3] [16] and references therein for the sources of these techniques.

Most of this section deals with the general case of Kähler reduction $M^4 \rightarrow \Sigma^2$. We do not specialize to Taub-NUTs or any other special metric families until Section 2.4.2.

2.1 Polytope construction

Because $\mathcal{L}_{\mathcal{X}_i}\omega = 0$, we may define functions φ^1, φ^2 by $d\varphi^i = -i\mathcal{X}_i\omega$. This gives gradient fields $\nabla\varphi^1, \nabla\varphi^2$ that commute, so define integrable leaves, which are Lagrangian submanifolds. Assigning to one leaf a value of $(0, 0)$ for (θ^1, θ^2) , then we can then define (θ^1, θ^2) functions on the entire manifold as push-forwards along the $\mathcal{X}_1, \mathcal{X}_2$ action. The construction gives the so-called *action-angle coordinates* $(\varphi^1, \varphi^2, \theta^1, \theta^2)$. The moment map is just forgetting the angle coordinates, and gives the *moment polytope*

$$\Sigma^2 \triangleq \text{Image}[(\varphi^1, \varphi^2) : M^4 \longrightarrow \mathbb{R}^2]. \quad (11)$$

This map is a submersion except where \mathcal{X}_1 or \mathcal{X}_2 have zeros or are colinear, and it is well-known that the image is in fact a polygon, which in general may or may not include all of its edges, and might be all of \mathbb{R}^2 . Because \mathcal{X}_1 or \mathcal{X}_2 are also Killing, Σ^2 inherits a Riemannian metric which is obviously smooth in the interior, and is in fact smooth at the boundary except at corners where it is Lipschitz. The polytope is naturally isometrically isomorphic with the completion of any of the $\{\nabla\varphi^1, \nabla\varphi^2\}$ leaves, and has

a natural complex structure (the Hodge star), which is not inherited from J on M^4 .

2.2 Metric quantities

One can show that the metrics, complex structures, and symplectic structures on M^4 and Σ^2 are

$$g_4 = \left(\begin{array}{c|c} G & 0 \\ \hline 0 & G^{-1} \end{array} \right), \quad J_4 = \left(\begin{array}{c|c} 0 & -G^{-1} \\ \hline G & 0 \end{array} \right), \quad \omega_4 = \left(\begin{array}{c|c} 0 & -Id \\ \hline Id & 0 \end{array} \right) \quad (12)$$

$$g_\Sigma = G, \quad J_\Sigma = \frac{1}{\sqrt{\mathcal{V}}} \left(\begin{array}{cc} \langle \mathcal{X}_1, \mathcal{X}_2 \rangle & -|\mathcal{X}_1|^2 \\ |\mathcal{X}_2|^2 & -\langle \mathcal{X}_1, \mathcal{X}_2 \rangle \end{array} \right), \quad (13)$$

where we have set

$$G = \mathcal{V}^{-1} \left(\begin{array}{cc} |\mathcal{X}_2|^2 & -\langle \mathcal{X}_1, \mathcal{X}_2 \rangle \\ -\langle \mathcal{X}_1, \mathcal{X}_2 \rangle & |\mathcal{X}_1|^2 \end{array} \right), \quad G^{-1} = \left(\begin{array}{cc} |\mathcal{X}_1|^2 & \langle \mathcal{X}_1, \mathcal{X}_2 \rangle \\ \langle \mathcal{X}_1, \mathcal{X}_2 \rangle & |\mathcal{X}_2|^2 \end{array} \right), \quad (14)$$

$$\mathcal{V} \triangleq \text{Det}(G^{-1}) = |\nabla \varphi^1|^2 |\nabla \varphi^2|^2 - \langle \nabla \varphi^1, \nabla \varphi^2 \rangle^2.$$

We remark that G is a Hessian, $G = \nabla^2 U$, for a function U known as that the *Kähler potential* [7] of M^4 , but we shall not pursue this further.

The φ^i are neither pluriharmonic nor even harmonic on M^4 or Σ^2 , but the functions θ^i are pluriharmonic on M^4 and so can be used to determine holomorphic coordinates (z^1, z^2) and a holomorphic framing of M^4 . It is possible to compute

$$\frac{\partial}{\partial z^1} = \frac{1}{2} (\nabla \varphi^1 - \sqrt{-1} \mathcal{X}_1), \quad \frac{\partial}{\partial z^2} = \frac{1}{2} (\nabla \varphi^2 - \sqrt{-1} \mathcal{X}_2) \quad (15)$$

$$dz^1 = Jd\theta^1 + \sqrt{-1}d\theta^1, \quad dz^2 = Jd\theta^2 + \sqrt{-1}d\theta^2$$

One computes the Hermitian metric $h_{i\bar{j}}$ to be just $\frac{1}{2}(G^{-1})_{ij}$, so $\det h_{i\bar{j}} = \frac{1}{4}\mathcal{V}$.

2.3 Curvature quantities

Because $\det(h_{i\bar{j}}) = \frac{1}{4}\mathcal{V}$, the Ricci form and scalar curvature of (M^4, J, ω) are

$$\rho = -2\sqrt{-1}\partial\bar{\partial} \log \mathcal{V} = dJd \log \mathcal{V}, \quad (16)$$

$$s = -\Delta \log \mathcal{V}.$$

The function s is $\mathcal{X}_1, \mathcal{X}_2$ invariant and so passes down to Σ^2 , where $s = -\Delta \log \mathcal{V}$ becomes

$$\Delta_\Sigma \sqrt{\mathcal{V}} + \frac{1}{2}s\sqrt{\mathcal{V}} = 0. \quad (17)$$

When $s = 0$ the function $x = \sqrt{\mathcal{V}}$ is an harmonic coordinate on Σ with harmonic conjugate y (the solution of $dy = -J_\Sigma dx$). It is not difficult to prove that, if the polytope boundary has one component, then the complex variable $z = x + iy$ has no critical points, so it is a global complex coordinate that maps Σ^2 onto the right half-plane. If the boundary has

no components, then by results in [16] the complex coordinate is constant, and Σ^2 and M^4 are flat Riemannian manifolds. If the polytope has more than one boundary component, z may or may not be a branched map to the right half-plane. These statements are proved in [16], where examples are also provided.

Letting $A = \left(\frac{\partial \varphi^i}{\partial x^j} \right)$ be the coordinate transition matrix, the metric and sectional curvature in x - y coordinates are

$$\begin{aligned} g_\Sigma &= \frac{\det(A)}{x} (dx \otimes dx + dy \otimes dy), \quad A = \left(\frac{\partial \varphi^i}{\partial x^j} \right), \\ K_\Sigma &= -\frac{x}{\det(A)} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \log \sqrt{\frac{\det(A)}{x}}. \end{aligned} \quad (18)$$

where $x^1 = x$, $x^2 = y$. The Σ^2 sectional curvature is indeed the sectional curvature of the totally geodesic $\{\nabla \varphi^1, \nabla \varphi^2\}$ leaves in M^4 . In order to determine all quantities on M^4 it suffices simply to express φ^1, φ^2 as functions of x, y .

Finally we consider the Ricci curvature of M^4 . This is invariant under $\mathcal{X}_1, \mathcal{X}_2$, so it stands to reason it is accessible from Σ^2 in some way. To see how, note that since J and $\log \mathcal{V}$ are invariant under the $\mathcal{X}_i = \frac{\partial}{\partial \theta^i}$, the fact that $\mathcal{L}_{\mathcal{X}_i} \rho = 0$ allows us to define potentials via

$$i \frac{\partial}{\partial \theta^i} \rho = -d \left(i \frac{\partial}{\partial \theta^i} J d \log \mathcal{V} \right) = d \langle \nabla \varphi^i, \nabla \log \mathcal{V} \rangle. \quad (19)$$

We call $\mathcal{R}_1 = -\langle \nabla \varphi^1, \nabla \log \mathcal{V} \rangle$ and $\mathcal{R}_2 = -\langle \nabla \varphi^2, \nabla \log \mathcal{V} \rangle$ the *Ricci potentials*. These are invariant functions so pass down to Σ^2 . On M^4 we clearly have $\rho = d\mathcal{R}_1 \wedge d\theta^1 + d\mathcal{R}_2 \wedge d\theta^2$. In the scalar-flat case, $*\rho = -\rho$ and so

$$|\rho|^2 dVol = -\rho \wedge \rho = d\mathcal{R}_1 \wedge d\mathcal{R}_2 \wedge d\theta^1 \wedge d\theta^2. \quad (20)$$

The 2-form $d\mathcal{R}_1 \wedge d\mathcal{R}_2$ makes sense on Σ^2 and is non-negative; we call it the *Ricci pseudo-volume form*. Unlike the potentials $\mathcal{R}_1, \mathcal{R}_2$, the pseudo-volume form is invariant under affine recombination of coordinates φ^1, φ^2 . Note that in the case $\mathcal{X}_1, \mathcal{X}_2$ actually define a toric structure (meaning their actions close to form a standard torus action), we have $\int_{T^2} d\theta^1 \wedge d\theta^2 = 4\pi^2$, so the total L^2 Ricci curvature energy can be expressed as an integral on Σ^2 :

$$L_{M^4}^2(|\text{Ric}|) = 4\pi^2 \int_{\Sigma^2} d\mathcal{R}_1 \wedge d\mathcal{R}_2. \quad (21)$$

2.4 The classification

2.4.1 The polytope outline

With $z = x + \sqrt{-1}y$, we have the bijective analytic map $z : \Sigma^2 \rightarrow \overline{H^2} \subset \mathbb{C}$, where $\partial \Sigma^2$ maps onto the y -axis. Thus we can study the variables φ^1, φ^2 as functions of (x, y) . It can be shown that these functions satisfy

$$x \left(\frac{\partial^2 \varphi^i}{\partial x^2} + \frac{\partial^2 \varphi^i}{\partial y^2} \right) - \frac{\partial \varphi^i}{\partial x} = 0. \quad (22)$$

This point of view was taken in [16], where it was shown that half-plane solutions of this non-uniformly elliptic equation are powerfully constrained. First, if $\varphi = 0$ on the y -axis, $\varphi \geq 0$, and φ solves (22), then $\varphi = Cx^2$ for some $C \geq 0$.

The *outline map* of the polytope is the map from the boundary of the half-plane to the φ^1, φ^2 plane, namely $(\varphi^1, \varphi^2) : \{x = 0\} \rightarrow \mathbb{R}^2$. It can be shown that this map is piecewise-linear, and in addition, given any outline map—any piecewise linear map from the y -axis to \mathbb{R}^2 that is convex and has finitely many facets—there are functions φ^1, φ^2 that map the right half-plane to the convex hull of the outline, and are equal to the outline map along the y -axis. Given the outline map, these functions can be written down *explicitly*, as was done in [5] [3] [16]. The Theorem 5.4 in [16] implies that if any other functions φ^1, φ^2 have this outline map, they each may differ from the explicit examples only by terms of the form Cx^2 where $C > 0$.

The upshot is that given any piecewise-linearly parametrized outline, there is *precisely* a 2-parameter family of functions φ^1, φ^2 that have the same outline, and that produce a positive definite metric by the recipe of equation (18).

2.4.2 Generalized Taub-NUTs and the exceptional Taub-NUT

This is the case that the polytope has one corner. After possible affine recombination of φ^1, φ^2 , the polytope is the first quadrant and the outline consists of the two unit-parametrized positive axes. The 2-parameter family of moment functions is

$$\varphi^1 = \frac{1}{\sqrt{2}} \left(-y + \sqrt{x^2 + y^2} \right) + \frac{\alpha}{2} x^2, \quad \varphi^2 = \frac{1}{\sqrt{2}} \left(y + \sqrt{x^2 + y^2} \right) + \frac{\beta}{2} x^2. \quad (23)$$

Setting $M = \frac{\alpha + \beta}{2}$ and $k = \frac{\alpha - \beta}{\alpha + \beta}$, we easily compute the polytope's metric and sectional curvature:

$$\begin{aligned} g_\Sigma &= \frac{1 + \sqrt{2}M \left(\sqrt{x^2 + y^2} + ky \right)}{\sqrt{x^2 + y^2}} (dx \otimes dx + dy \otimes dy) \\ K_\Sigma &= M \frac{-1 + \sqrt{2}Mk \left(y + k\sqrt{x^2 + y^2} \right)}{\left(1 + \sqrt{2}M \left(ky + \sqrt{x^2 + y^2} \right) \right)^3} \end{aligned} \quad (24)$$

using equations (18) above. Changing M simply scales the metric (one scales the coordinates simultaneously), and choosing $M = 0$ gives the flat metric. The parameter $k \in [-1, 1]$, the chirality number, changes the metric structure while leaving, say, $K_\Sigma(0, 0)$ equal to $-M$, so it does no scaling.

When $k = 1$, we see that the negative y -axis retains $K_\Sigma(0, -y) = -M$, so there is no curvature fall-off at all along that ray. Choosing $k = -1$ does the same with the positive y -axis, but all other choices of k give metrics with definite curvature fall-off along all paths to infinity.

2.4.3 Half-plane metrics

Next we consider the case that the polytope is the half-plane. After possible affine recombination the classification says we must have

$$\varphi^1 = x^2, \quad \varphi^2 = y + \frac{M}{2}yx^2 \quad (25)$$

for some $M > 0$. This gives polytope metric and sectional curvature

$$g_\Sigma = \left(1 + \frac{M}{2}x^2\right)(dx \otimes dx + dy \otimes dy), \quad K_\Sigma = -M \frac{1 - \frac{M}{2}x^2}{\left(1 + \frac{M}{2}x^2\right)^3} \quad (26)$$

Replacing x, y by $\tilde{x} = \sqrt{\frac{M}{2}}x, \tilde{y} = \sqrt{\frac{M}{2}}y$, gives $g_\Sigma = \frac{2}{M}(1 + \tilde{x}^2)(d\tilde{x}^2 + d\tilde{y}^2)$, so once again we see M is a scale parameter. The value $M = 0$ gives the flat metric, and $M \neq 0$ gives scaled versions of this instanton.

2.4.4 Planar metrics

If the polytope is complete, its metric is flat.

3 The generalized Taub-NUT metrics

As functions of the volumetric isothermal coordinates (x, y) , the (φ^1, φ^2) map takes H^2 to Σ^2 , where $H^2 = \{x > 0\}$ is the right half-plane in \mathbb{C} . For the generalized Taub-NUT metrics, the moment functions are

$$\varphi^1 = \frac{1}{\sqrt{2}}(-y + \sqrt{x^2 + y^2}) + \frac{\alpha}{2}x^2, \quad \varphi^2 = \frac{1}{\sqrt{2}}(y + \sqrt{x^2 + y^2}) + \frac{\beta}{2}x^2. \quad (27)$$

Setting $M = \frac{\alpha+\beta}{2}$ and $k = \frac{\alpha-\beta}{\alpha+\beta}$, and letting $r = \sqrt{x^2 + y^2}$, $\tan \theta = y/x$ be the usual polar coordinates on the half-plane $H^2 \subset \mathbb{C}$, we have polytope metric and sectional curvature

$$g_\Sigma = \frac{1 + \sqrt{2}Mr(1 + k \sin \theta)}{r}(dr^2 + r^2 d\theta^2), \quad 0 \leq r < \infty, \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2},$$

$$K_\Sigma = M \frac{-1 + \sqrt{2}Mkr(k + \sin \theta)}{(1 + \sqrt{2}Mr(1 + k \sin \theta))^3}. \quad (28)$$

In this section, it will be necessary to enforce $k \neq 1, -1$.

3.1 Coordinates

From x, y coordinates, we obtain u, v coordinates:

$$u = \sqrt{\frac{M}{\sqrt{2}}}\sqrt{\sqrt{x^2 + y^2} + y}, \quad v = \sqrt{\frac{M}{\sqrt{2}}}\sqrt{\sqrt{x^2 + y^2} - y}. \quad (29)$$

These are isothermal, and in fact are a complex square root of the (x, y) coordinates; we call then adapted quadratic coordinates. The image of

(u, v) from Σ^2 to \mathbb{C} is once again the first quadrant, not the right half-plane. The inverse transformation are $x = \frac{\sqrt{2}}{M}uv$, $y = \frac{1}{\sqrt{2}M}(u^2 - v^2)$. The moment functions and metric are

$$\begin{aligned}\varphi^1 &= \frac{v^2}{M} (1 + (1+k)u^2), \quad \varphi^2 = \frac{u^2}{M} (1 + (1-k)v^2), \\ g_\Sigma &= \frac{2\sqrt{2}}{M} (1 + (1+k)u^2 + (1-k)v^2) (du^2 + dv^2).\end{aligned}\tag{30}$$

On M^4 the functions u, v are not harmonic. Later we shall require use of the metric and complex structure in u, v, θ^1, θ^2 coordinates. These are

$$\begin{aligned}g &= \left(\frac{g_\Sigma}{G^{-1}} \right), \\ G^{-1} &= \frac{\sqrt{2}}{M} \begin{pmatrix} \frac{v^2(1+2(1+k)u^2+(1+k)^2u^2(u^2+v^2))}{1+(1+k)u^2+(1-k)v^2} & \frac{u^2v^2(2+(1-k^2)(u^2+v^2))}{1+(1+k)u^2+(1-k)v^2} \\ \frac{u^2v^2(2+(1-k^2)(u^2+v^2))}{1+(1+k)u^2+(1-k)v^2} & \frac{u^2(1+2(1-k)v^2+(1-k)^2v^2(u^2+v^2))}{1+(1+k)u^2+(1-k)v^2} \end{pmatrix}\end{aligned}\tag{31}$$

3.2 Distance functions and geodesic normal coordinates

3.2.1 The distance functions S_η

The form of g_Σ in (30) allows a separation of variables technique for finding certain solutions of $|\nabla S| = 1$. Supposing $S(u, v) = f(u) + h(v)$ and choosing some parameter $\eta \in [0, \pi/2]$, we have

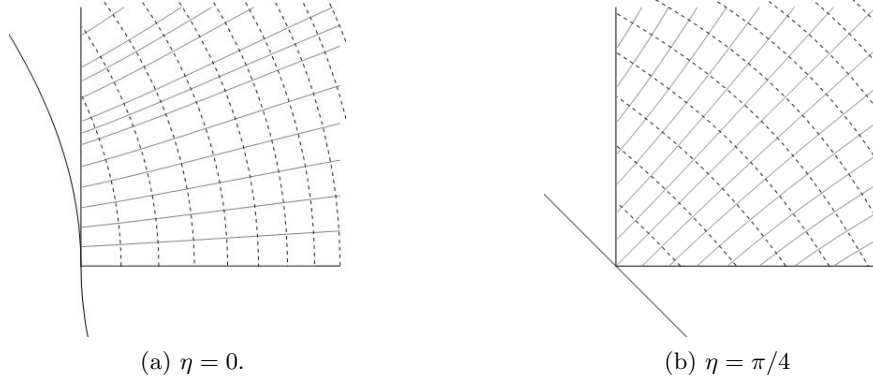
$$\begin{aligned}1 &= \frac{M}{2\sqrt{2}} \frac{(f_u)^2 + (h_v)^2}{(\cos^2 \eta + (1+k)u^2) + (\sin^2 \eta + (1-k)v^2)}, \\ \frac{df}{du} &= \sqrt{\frac{2\sqrt{2}}{M}} \sqrt{\cos^2 \eta + (1+k)u^2}, \quad \frac{dh}{dv} = \sqrt{\frac{2\sqrt{2}}{M}} \sqrt{\sin^2 \eta + (1-k)v^2}.\end{aligned}\tag{32}$$

Emphasizing the role of $\eta \in [0, \pi/2]$ we write $S = S_\eta$. Solving for f, h under the condition $S_\eta(0, 0) = 0$ gives

$$\begin{aligned}S_\eta(u, v) &= \sqrt{\frac{2\sqrt{2}}{M}} \frac{\cos^2 \eta}{2\sqrt{1+k}} \left[\frac{u\sqrt{1+k}}{\cos \eta} \sqrt{1 + \left(\frac{u\sqrt{1+k}}{\cos \eta} \right)^2} + \log \left(\frac{u\sqrt{1+k}}{\cos \eta} + \sqrt{1 + \left(\frac{u\sqrt{1+k}}{\cos \eta} \right)^2} \right) \right] \\ &\quad + \sqrt{\frac{2\sqrt{2}}{M}} \frac{\sin^2 \eta}{2\sqrt{1-k}} \left[\frac{v\sqrt{1-k}}{\cos \eta} \sqrt{1 + \left(\frac{v\sqrt{1-k}}{\cos \eta} \right)^2} + \log \left(\frac{v\sqrt{1-k}}{\cos \eta} + \sqrt{1 + \left(\frac{v\sqrt{1-k}}{\cos \eta} \right)^2} \right) \right]\end{aligned}\tag{33}$$

As depicted in Figure 1, the distance function S_η is not the distance to any locus within the polytope, but to some virtual $S_\eta = 0$ curve in the u, v plane that intersects the polytope only at $(0, 0)$.

Figure 1: Contour plots of the distance function S_η for two values of η . We have chosen $k = 0.5$. The solid curve is the virtual locus $S_\eta = 0$, which touches the polytope only at $(0,0)$. Dashed lines are additional level-sets, and thin solid curves are characteristic curves for S_η , which are geodesics. One characteristic intersects the origin for each choice of η .



3.2.2 The geodesics based at the origin

Each choice of η allows us to find a single geodesic from the origin, which, as we shall see, is the geodesic that makes an angle of η with the u -axis. We have

$$\nabla S_\eta = \sqrt{\frac{2\sqrt{2}}{M}} \left(\frac{\sqrt{\cos^2 \eta + (1+k)u^2}}{1 + (1+k)u^2 + (1-k)v^2} \frac{\partial}{\partial u} + \frac{\sqrt{\sin^2 \eta + (1-k)v^2}}{1 + (1+k)u^2 + (1-k)v^2} \frac{\partial}{\partial v} \right) \quad (34)$$

so that if $\gamma(t) = (u(t), v(t))$ is a characteristic path, meaning $\frac{d\gamma}{dt} = \nabla S_\eta$, then γ is a geodesic, and solves the coupled autonomous system

$$\frac{du}{dt} = \frac{\sqrt{\cos^2 \eta + (1+k)u^2}}{1 + (1+k)u^2 + (1-k)v^2}, \quad \frac{dv}{dt} = \frac{\sqrt{\sin^2 \eta + (1-k)v^2}}{1 + (1+k)u^2 + (1-k)v^2}. \quad (35)$$

This is still difficult to solve, but eliminating the t variable gives

$$\frac{dv}{du} = \frac{\sqrt{\sin^2 \eta + (1-k)v^2}}{\sqrt{\cos^2 \eta + (1+k)u^2}} \quad (36)$$

which separates. At $(u, v) = (0, 0)$ we have $\frac{dv}{du} = \tan \eta$, which gives η its geometric meaning. The solution for initial condition $\gamma(0) = (0, 0)$ is given explicitly by the following equation, the *unparametrized geodesic equation* for geodesics emanating from the origin:

$$\left(V_\eta + \sqrt{1 + V_\eta} \right)^{\frac{1}{\sqrt{1-k}}} = \left(U_\eta + \sqrt{1 + U_\eta} \right)^{\frac{1}{\sqrt{1+k}}} \quad (37)$$

where we have set $U_\eta = \frac{\sqrt{1+k}u}{\cos \eta}$, $V_\eta = \frac{\sqrt{1-k}v}{\sin \eta}$. Solve for v in terms of u even more explicitly,

$$v = \frac{\sin \eta}{2\sqrt{1-k}} \left[\left(\frac{u\sqrt{1+k}}{\cos \eta} + \sqrt{1 + \left(\frac{u\sqrt{1+k}}{\cos \eta} \right)^2} \right)^{\sqrt{\frac{1-k}{1+k}}} - \left(-\frac{u\sqrt{1+k}}{\cos \eta} + \sqrt{1 + \left(\frac{u\sqrt{1+k}}{\cos \eta} \right)^2} \right)^{\sqrt{\frac{1-k}{1+k}}} \right]. \quad (38)$$

Values of η between 0 and $\pi/2$ indeed give all geodesics from the origin.

3.2.3 Geodesic Normal Coordinates

Let $R = \text{dist}(o, \cdot)$ be the distance function to the origin. If (u, v) is an arbitrary point in the first quadrant, we wish to find both $R(u, v)$ and $\eta(u, v)$, where η is the angle at the origin of the geodesic to (u, v) . That is, given (u, v) , we must solve the geodesic equation (37) for η :

$$\left(\frac{u\sqrt{1+k}}{\cos \eta} + \sqrt{1 + \left(\frac{u\sqrt{1+k}}{\cos \eta} \right)^2} \right)^{\frac{1}{\sqrt{1+k}}} = \left(\frac{v\sqrt{1-k}}{\sin \eta} + \sqrt{1 + \left(\frac{v\sqrt{1-k}}{\sin \eta} \right)^2} \right)^{\frac{1}{\sqrt{1-k}}} \quad (39)$$

which of course is a non-constructive step (we omit the straightforward proof that there is a unique solution in $[0, \pi/2]$). Having found $\eta = \eta(u, v)$, then to find $R = R(u, v)$, one simply has

$$R = S_{\eta(u,v)}(u, v). \quad (40)$$

Thus we have described the transition from the isothermal system (u, v) to polar geodesic coordinates (R, η) , as depicted in Figure 2.

The reverse transformation is more important. Given initial angle η and distance R , we must find (u, v) . This is equivalent to finding the parametrization for the geodesics described by (37). We must solve the nonlinear non-algebraic system

$$\left(U_\eta + \sqrt{1 + U_\eta^2} \right)^{\frac{1}{\sqrt{1+k}}} = \left(V_\eta + \sqrt{1 + V_\eta^2} \right)^{\frac{1}{\sqrt{1-k}}}, \quad S_\eta(u, v) = R \quad (41)$$

for (u, v) . To do so explicitly, define the auxiliary function F by

$$F \triangleq \left(U_\eta + \sqrt{1 + U_\eta^2} \right)^{\frac{1}{\sqrt{1+k}}} = \left(V_\eta + \sqrt{1 + V_\eta^2} \right)^{\frac{1}{\sqrt{1-k}}} \quad (42)$$

and note that the second equation of (41) forces F to be a function of R and η alone, due to the expression of S_η in (33). That is, one must solve the following equation for F :

$$\begin{aligned} & \frac{\cos^2 \eta}{2\sqrt{1+k}} \left[\frac{1}{4} \left(F^{2\sqrt{1+k}} - F^{-2\sqrt{1+k}} \right) + \log F^{\sqrt{1+k}} \right] \\ & + \frac{\sin^2 \eta}{2\sqrt{1-k}} \left[\frac{1}{4} \left(F^{2\sqrt{1-k}} - F^{-2\sqrt{1-k}} \right) + \log F^{\sqrt{1-k}} \right] = \sqrt{\frac{M}{2\sqrt{2}}} R \end{aligned} \quad (43)$$

to obtain $F = F(\eta, R)$. This is the non-constructive (that is to say transcendental) step. To see that solutions are unique, note that for any choice of η the left-hand side of (43) is monotone-increasing in F . Then (42) gives

$$\begin{aligned} u(R, \eta) &= \frac{\cos \eta}{2\sqrt{1+k}} \left(F^{\sqrt{1+k}} - F^{-\sqrt{1+k}} \right) \\ v(R, \eta) &= \frac{\sin \eta}{2\sqrt{1-k}} \left(F^{\sqrt{1-k}} - F^{-\sqrt{1-k}} \right). \end{aligned} \quad (44)$$

See Section 3.2.4 for a simple way of approximating u , v , and F as functions of R and η to arbitrary accuracy (for sufficiently large R).

To compute the metric in geodesic polar coordinates, consider again the unparametrized geodesic equation (37), which relates u , v , and η . Linearizing gives

$$\frac{du + u \tan \eta d\eta}{\sqrt{\cos^2 \eta + (1+k)u^2}} = \frac{dv - v \cot \eta d\eta}{\sqrt{\sin^2 \eta + (1-k)v^2}}, \quad (45)$$

and then solving for $d\eta$ and norming gives

$$|d\eta|^2 \frac{2\sqrt{2}}{M} \left(u \tan \eta \sqrt{\sin^2 \eta + (1-k)v^2} + v \cot \eta \sqrt{\cos^2 \eta + (1+k)u^2} \right) = 1. \quad (46)$$

Using (44) to write $|d\eta|^2$ in terms of F , η , we obtain, finally, the polytope metric in geodesic polar coordinates:

$$\begin{aligned} g_\Sigma &= dR \otimes dR + A(R, \eta)^2 d\eta \otimes d\eta, \quad \text{where} \\ A(R, \eta)^2 &= |d\eta|^{-2} \\ &= \frac{\sin^2 \eta}{\sqrt{2}M\sqrt{1+k}} \left(F^{\sqrt{1+k}} - F^{-\sqrt{1+k}} \right) \left(F^{\sqrt{1-k}} + F^{-\sqrt{1-k}} \right) \\ &\quad + \frac{\cos^2 \eta}{\sqrt{2}M\sqrt{1-k}} \left(F^{\sqrt{1+k}} + F^{-\sqrt{1+k}} \right) \left(F^{\sqrt{1-k}} - F^{-\sqrt{1-k}} \right). \end{aligned} \quad (47)$$

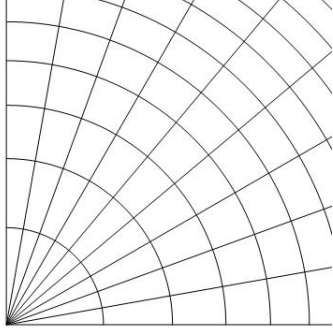
3.2.4 Asymptotic approximations of functions

We can approximate the value of F to within tolerable margins by

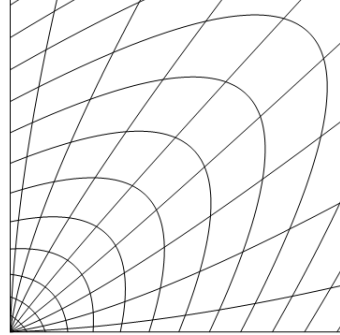
$$\tilde{F}(R, \eta) = \begin{cases} \left(\frac{8\sqrt{1+k}}{\cos^2 \eta} \sqrt{\frac{M}{2\sqrt{2}}} R \right)^{\frac{1}{2\sqrt{1+k}}}, & 0 \leq \eta < \sin^{-1} \left(\frac{(\sqrt{\frac{M}{2\sqrt{2}}} R)^{\sqrt{\frac{1+k}{1-k}}-1}}{(\sqrt{\frac{M}{2\sqrt{2}}} R)^{\sqrt{\frac{1+k}{1-k}}-1} + \frac{3}{4} \frac{8\sqrt{1+k}}{(8\sqrt{1-k})\sqrt{\frac{1+k}{1-k}}}} \right) \\ \left(\frac{8\sqrt{1-k}}{\sin^2 \eta} \sqrt{\frac{M}{2\sqrt{2}}} R \right)^{\frac{1}{2\sqrt{1-k}}}, & \sin^{-1} \left(\frac{(\sqrt{\frac{M}{2\sqrt{2}}} R)^{\sqrt{\frac{1+k}{1-k}}-1}}{(\sqrt{\frac{M}{2\sqrt{2}}} R)^{\sqrt{\frac{1+k}{1-k}}-1} + \frac{3}{4} \frac{8\sqrt{1+k}}{(8\sqrt{1-k})\sqrt{\frac{1+k}{1-k}}}} \right) \leq \eta \leq \frac{\pi}{2}. \end{cases} \quad (48)$$

where “tolerable margins” means the following:

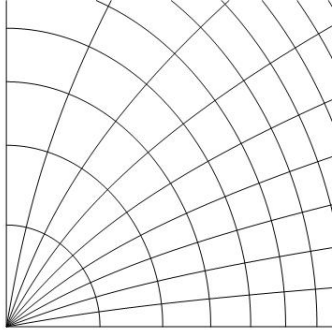
Figure 2: *Geodesic polar coordinates in adapted quadratic coordinates and in momentum coordinates. Shown are radial geodesics from the origin, and evenly spaced level-sets of the distance function.*



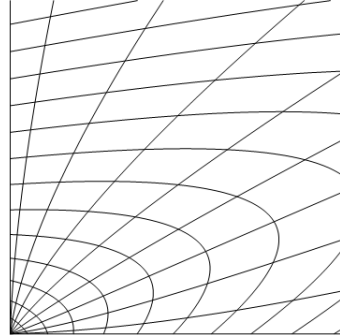
(a) (u, v) coordinates, $k = 0$.



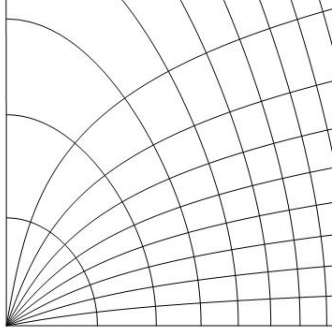
(b) (φ^1, φ^2) coordinates, $k = 0$.



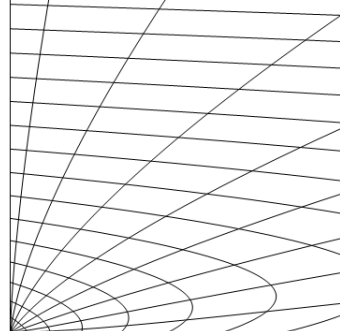
(c) (u, v) coordinates, $k = 0.5$.



(d) (φ^1, φ^2) coordinates, $k = 0.5$.



(e) (u, v) coordinates, $k = 1$.



(f) (φ^1, φ^2) coordinates, $k = 1$.

Lemma 3.1 Define the function $\mathcal{R} = \mathcal{R}(X, \eta)$ by

$$\begin{aligned} \mathcal{R} = & \sqrt{\frac{2\sqrt{2}}{M} \frac{\cos^2 \eta}{2\sqrt{1+k}}} \left[\frac{1}{4} \left(X^{2\sqrt{1+k}} - X^{-2\sqrt{1+k}} \right) + \log X^{\sqrt{1+k}} \right] \\ & + \sqrt{\frac{2\sqrt{2}}{M} \frac{\sin^2 \eta}{2\sqrt{1-k}}} \left[\frac{1}{4} \left(X^{2\sqrt{1-k}} - X^{-2\sqrt{1-k}} \right) + \log X^{\sqrt{1-k}} \right], \end{aligned} \quad (49)$$

so that if $F = F(R, \eta)$ solves (43) then of course $\mathcal{R}(F, \eta) = R$. Given any $\epsilon > 0$, then for sufficiently large R we have

$$\frac{1}{R} \mathcal{R}(\tilde{F}(R, \eta), \eta) = \frac{\mathcal{R}(\tilde{F}(R, \eta), \eta)}{\mathcal{R}(F(R, \eta), \eta)} \in [1, 2 + \epsilon] \quad (50)$$

for all $\eta \in [0, \pi/2]$.

Proof. Apply the first derivative test in the parameter η to learn that the minimum of $\frac{1}{R} \mathcal{R}(\tilde{F}(R, \eta), \eta)$ occurs at the endpoints $\eta = 0, \pi/2$, and that the maximum occurs at the discontinuity point, where the left and right limits are different. Then test these points to learn that the minimum is 1 and the maximum is a bit bigger than 2. \square

In short, our approximation \tilde{F} for F gives the correct value of R to within about a factor of 2, and further, this estimate is uniform in η . Also, for a fixed η and sufficiently large R , a single iteration of Newton's method improves this approximation arbitrarily closely to the true value, but this is not uniformly with respect to η . For large R , just one iteration of Hausholder's method with exponent $\sqrt{\frac{1+k}{1-k}}$ will make this uniform in η .

Next we explore a close approximation for the value of the distance function R in terms of (u, v) . Equation (33) provides one relationship between (R, η) and (u, v) , which can be written

$$\begin{aligned} \sqrt{\frac{M}{2\sqrt{2}}} \cdot R = & \frac{1}{2\sqrt{1+k}} \cos^2 \eta \left[\frac{\sqrt{1+ku}}{\cos \eta} \sqrt{1 + \left(\frac{\sqrt{1+ku}}{\cos \eta} \right)^2} \right. \\ & \left. + \log \left(\frac{\sqrt{1+ku}}{\cos \eta} + \sqrt{1 + \left(\frac{\sqrt{1+ku}}{\cos \eta} \right)^2} \right) \right] \\ & + \frac{1}{2\sqrt{1-k}} \sin^2 \eta \left[\frac{\sqrt{1-kv}}{\sin \eta} \sqrt{1 + \left(\frac{\sqrt{1-kv}}{\sin \eta} \right)^2} \right. \\ & \left. + \log \left(\frac{\sqrt{1-kv}}{\sin \eta} + \sqrt{1 + \left(\frac{\sqrt{1-kv}}{\sin \eta} \right)^2} \right) \right]. \end{aligned} \quad (51)$$

The other relationship, giving η in terms of (u, v) , is the geodesic equation (37).

Lemma 3.2 (Almost Spheres) Assume $\tilde{R} > 0$, and consider the locus $AS(\tilde{R})$

$$= \left\{ (u, v) \mid u = \sqrt[4]{\frac{\sqrt{2}M}{1+k}} \sqrt{\tilde{R}} \cos \psi, v = \sqrt[4]{\frac{\sqrt{2}M}{1-k}} \sqrt{\tilde{R}} \sin \psi, \psi \in [0, \pi/2] \right\} \quad (52)$$

which we call the “almost-sphere” of radius \tilde{R} . There are constants $\underline{C}_1, \underline{C}_2, \overline{C}_1, \overline{C}_2, \epsilon$ so that if (u, v) is any point on $AS(\tilde{R})$, we have

$$R + \underline{C}_1 \log R + \underline{C}_2 \leq \tilde{R} \leq (1 + \epsilon)R + \overline{C}_1 \log R + \overline{C}_2 \quad (53)$$

where $R = R(u, v)$ is the distance function to the origin, the $\underline{C}_1, \underline{C}_2, \overline{C}_1, \overline{C}_2$ depend only on M, k , and $\epsilon = \epsilon(M, \tilde{R})$ gets arbitrarily small as \tilde{R} increases

Proof. Choose any $\tilde{R} > 0$, and set $u = \sqrt[4]{\frac{\sqrt{2}M}{1+k}} \sqrt{\tilde{R}} \cos \psi$, $v = \sqrt[4]{\frac{\sqrt{2}M}{1+k}} \sqrt{\tilde{R}} \sin \psi$ so the values $\psi \in [0, \pi/2]$ sweep out the almost-sphere. Let $\eta = \eta(u, v)$ be the angle defined implicitly by (37). Then (51) lets us compute the distance R from the origin to (u, v) . For convenience we set $M = \frac{1}{\sqrt{2}}$, while keeping in mind that both R and \tilde{R} scale like $1/\sqrt{M}$. We have

$$\begin{aligned} R &= \tilde{R} \left(\cos \psi \sqrt{\frac{\cos^2 \eta}{\tilde{R}\sqrt{1+k}} + \cos^2 \psi} + \sin \psi \sqrt{\frac{\sin^2 \eta}{\tilde{R}\sqrt{1-k}} + \sin^2 \psi} \right) \\ &+ \frac{\cos^2 \eta}{\sqrt{1+k}} \log \left(\frac{\sqrt[4]{1+k} \sqrt{\tilde{R}} \cos \psi + \sqrt{\cos^2 \eta + \sqrt{1+k} \tilde{R} \cos^2 \psi}}{\cos \eta} \right) \\ &+ \frac{\sin^2 \eta}{\sqrt{1-k}} \log \left(\frac{\sqrt[4]{1-k} \sqrt{\tilde{R}} \sin \psi + \sqrt{\sin^2 \eta + \sqrt{1-k} \tilde{R} \sin^2 \psi}}{\sin \eta} \right) \end{aligned} \quad (54)$$

A simple approximation for large \tilde{R} shows that the non-logarithmic terms sum to just larger than 1, and the logarithmic terms are bounded above and below by fixed multiples of $\log \tilde{R}$. Thus indeed

$$R \in \left(\tilde{R} + \underline{C}_1 \log \tilde{R} + \underline{C}_2, (1 + \epsilon)\tilde{R} + \overline{C}_1 \log \tilde{R} + \overline{C}_2 \right). \quad (55)$$

□

Corollary 3.3 (The Almost Distance Function \tilde{R}) Define $\tilde{R} = \tilde{R}(u, v)$ by $\tilde{R} = \frac{1}{\sqrt{\sqrt{2}M}} ((1+k)u^2 + (1-k)v^2)$. Then there are functions $0 < \underline{\epsilon}(R) < \bar{\epsilon}(R)$, with $\lim_{R \rightarrow \infty} \underline{\epsilon}(R) = \lim_{R \rightarrow \infty} \bar{\epsilon}(R) = 0$ so that

$$R(1 + \underline{\epsilon}(R)) < \tilde{R} < R(1 + \bar{\epsilon}(R)). \quad (56)$$

Proof. Obvious from Lemma 3.2. Note $\underline{\epsilon}(R)$ and $\bar{\epsilon}(R)$ are roughly $R^{-1} \log R$. □

3.3 Computation of the key asymptotic ratios

Our primary tool will be the “almost polar coordinates” $\tilde{R}, \psi, \theta^1, \theta^2$, where the transitions are $u = \sqrt[4]{\frac{\sqrt{2}M}{1+k}} \sqrt{\tilde{R}} \cos \psi$, $v = \sqrt[4]{\frac{\sqrt{2}M}{1-k}} \sqrt{\tilde{R}} \sin \psi$. To justify the phrase “almost polar coordinates,” note that from (56) we have $\tilde{R}(1 + \underline{\epsilon}(R)) < \tilde{R} < R(1 + \bar{\epsilon}(R))$ (however the angle parameters ψ and η cannot be uniformly bounded in terms of one another).

3.3.1 Asymptotic Volume Growth

Let $B(R)$ be the ball of radius R centered at the origin, and let $AB(R)$ be the “almost-ball” of radius R about the origin. Specifically, we set

$$AB(R) = \left\{ \begin{aligned} u &= \sqrt[4]{\frac{\sqrt{2}M}{1+k}} \sqrt{\tilde{R}} \cos \psi, \\ v &= \sqrt[4]{\frac{\sqrt{2}M}{1+k}} \sqrt{\tilde{R}} \sin \psi \mid 0 \leq \tilde{R} \leq R, 0 \leq \psi \leq \pi/2 \end{aligned} \right\}. \quad (57)$$

By (55), we have $AB(R) \subset B(R) \subset AB(R(1+\epsilon))$, for some ϵ with $\lim_{R \rightarrow \infty} \epsilon = 0$, and therefore $\text{Vol } B(R) < \text{Vol } AB(R) < \text{Vol } B(R(1+\epsilon(R)))$.

Proposition 3.4 *If $k \in (-1, 1)$, then asymptotic volume growth of balls is cubic: $\text{Vol } B(R) = O(R^3)$.*

Proof. In u, v, θ^1, θ^2 coordinates the volume form is

$$d\text{Vol} = \frac{4}{M^2} (1 + (1+k)u^2 + (1-k)v^2) uv du \wedge dv \wedge d\theta^1 \wedge d\theta^2. \quad (58)$$

Transitioning to almost polar coordinates, with transitions $u = \sqrt[4]{\frac{\sqrt{2}M}{1+k}} \sqrt{\tilde{R}} \cos \psi$, $v = \sqrt[4]{\frac{\sqrt{2}M}{1+k}} \sqrt{\tilde{R}} \sin \psi$, we obtain

$$\begin{aligned} d\text{Vol} &= \frac{2\sqrt{2}}{M} \left(1 + \sqrt{1+k} \sqrt{\sqrt{2}M} \tilde{R} \cos^2 \psi + \sqrt{1-k} \sqrt{\sqrt{2}M} \tilde{R} \sin^2 \psi \right) \\ &\quad \cdot \frac{\tilde{R} \cos \psi \sin \psi}{\sqrt{(1+k)(1-k)}} d\tilde{R} \wedge d\psi \wedge d\theta^1 \wedge d\theta^2 \end{aligned} \quad (59)$$

Integrating ψ, θ^1, θ^2 through their maximum ranges and integrating \tilde{R} from 0 to R , we obtain

$$\begin{aligned} \text{Vol}(AB(R)) &= \frac{2\sqrt{2}\pi^2}{M\sqrt{(1+k)(1-k)}} \left(R^2 + \frac{1}{3}\sqrt{1+k}\sqrt{\sqrt{2}M}R^3 + \frac{1}{3}\sqrt{1-k}\sqrt{\sqrt{2}M}R^3 \right). \end{aligned} \quad (60)$$

Therefore using Lemma 3.2 to approximate balls with almost-balls, we have

$$\begin{aligned} \text{Vol } B(R) &\leq R^3 \frac{2\sqrt{2}\pi^2}{M\sqrt{(1+k)(1-k)}} \left(R^{-1} + \frac{1}{3}\sqrt{1+k}\sqrt{\sqrt{2}M} + \frac{1}{3}\sqrt{1-k}\sqrt{\sqrt{2}M} \right) \\ &\leq \text{Vol } B(R \cdot (1+\epsilon(R))) \end{aligned} \quad (61)$$

so we see that volume growth is indeed cubic. Expressing this more simply,

$$C_1(k, M)R^3(1-\epsilon(R))^3 \leq \text{Vol}(B(R)) \leq C_2(k, M)R^3. \quad (62)$$

Therefore each instanton, except possibly the exceptional cases $k = \pm 1$, has precisely cubic volume growth. \square

3.3.2 Asymptotic Sectional Curvature, and energy computation

A simple algebraic fact is that on a Kähler 4-manifold with zero scalar curvature, all sectional curvatures are determined by the Ricci curvature and one single sectional curvature. We have already computed K_Σ in (28).

Lemma 3.5 *If K_Σ is the polytope sectional curvature and $k \notin \{-1, 0, 1\}$, then $K_\Sigma = O(R^{-2})$, except along a single geodesic, where $K_\Sigma = O(R^{-3})$. If $k = 0$ then $K_\Sigma = O(R^{-3})$ along all geodesics.*

Proof. In (u, v) coordinates we have

$$K_\Sigma = M \frac{-1 + k(u^2 + v^2)(k + \cos 2\eta)}{(1 + (u^2 + v^2)(1 + k \cos 2\eta))^3}. \quad (63)$$

Using the approximation (56) and the transitions $u = \sqrt[4]{\frac{\sqrt{2}M}{1+k}} \sqrt{\tilde{R}} \cos \psi$, $v = \sqrt[4]{\frac{\sqrt{2}M}{1-k}} \sqrt{\tilde{R}} \sin \psi$, we have $u^2 + v^2 = \tilde{R} \sqrt{\sqrt{2}M} \left(\frac{\cos^2 \psi}{\sqrt{1+k}} + \frac{\sin^2 \psi}{\sqrt{1-k}} \right)$. Therefore (63) clearly gives $K_\Sigma = O(\tilde{R}^{-2}) = O(R^{-2})$ except along a single geodesic where $K_\Sigma = O(R^{-3})$.

The exceptions are $k = 0$ and $k = \pm 1$. When $k = 0$ we see that $K_\Sigma = O(R^{-3})$ everywhere. When $k = \pm 1$ we have that $K_\Sigma = O(R^{-3})$ everywhere except on either the u or v axis, where sectional curvature does not decay at all: $K_\Sigma = -M$. \square

Proposition 3.6 (Chirality number and L^2 norms) *The L^2 norms of the Ricci and Riemann tensors are*

$$\begin{aligned} L^2(|\text{Ric}|) &= \frac{4\pi^2 k^2}{1 - k^2} \\ L^2(|\text{Rm}|) &= \frac{16\pi^2(2 - k^2)}{1 - k^2} \geq 32\pi^2. \end{aligned} \quad (64)$$

Proof. We compute the Ricci pseudo-volume on Σ^2 and the total L^2 Ricci curvature. The two Ricci potentials are easily computed:

$$\begin{aligned} \mathcal{R}_1 &= \langle \nabla \log x, \nabla \varphi^1 \rangle = \frac{1}{\sqrt{2}} \frac{1 + (1+k)(u^2 + v^2)}{1 + (1+k)u^2 + (1-k)v^2} \\ \mathcal{R}_2 &= \langle \nabla \log x, \nabla \varphi^2 \rangle = \frac{1}{\sqrt{2}} \frac{1 + (1-k)(u^2 + v^2)}{1 + (1+k)u^2 + (1-k)v^2} \end{aligned} \quad (65)$$

and the Ricci pseudo-volume is equally easily computed:

$$d\mathcal{R}_1 \wedge d\mathcal{R}_2 = \frac{8k^2 uv}{(1 + (1+k)u^2 + (1-k)v^2)^3} du \wedge dv. \quad (66)$$

Therefore, according to the method of Section 2.3, the L^2 norm of Ricci curvature is

$$\begin{aligned} \int_{M^4} |\text{Ric}|^2 d\text{Vol} &= 4\pi^2 \int_{\Sigma^2} \frac{8k^2 uv}{(1 + (1+k)u^2 + (1-k)v^2)^3} du \wedge dv \\ &= \frac{4\pi^2 k^2}{1 - k^2}. \end{aligned} \quad (67)$$

To compute the L^2 norm of the Riemann tensor, we use the Chern-Gauss-Bonnet integral for scalar-flat manifolds

$$32\pi^2\chi(M^4) = \int (|\text{Rm}|^2 - 4|\text{Ric}|^2) dVol + \lim_{R \rightarrow \infty} \int_{\partial B(R)} \mathcal{TP}_\chi \quad (68)$$

where the boundary term \mathcal{TP}_χ is the Chern-Simons 3-form. The norm of this 3-form can be estimated by $|\mathcal{TP}_\chi| < C_1|\text{Rm}||II| + C_2|II|^3$ where II is the second fundamental form of the outer boundary (whether it be a geodesic sphere or perhaps an almost-sphere).

If R is the distance function and the metric g is expressed in coordinates that are compatible with R , then II can be expressed as derivatives of metric components: $II_{ij} = \nabla_R(g_{ij})$. But from 31 it is not difficult (using computer technology perhaps) to compute that generically $|II| = O(R^{-1})$. Likewise, in the next proposition below, we compute that $|\text{Rm}| = O(R^{-2})$ (unless $k = 0$ where $|\text{Rm}| = O(R^{-3})$). Thus

$$\left| \int_{\partial B(R)} \mathcal{TP}_\chi \right| \leq CR^{-3} \cdot \text{Area}(\partial B(R)). \quad (69)$$

It is possible to determine the asymptotic area ratio directly, using approximation methods similar to those in Section 3.3.1. Although such estimates are conceptually simple they are technically complicated. Instead, we simply note that the limit in (68) may be taken along any subsequence, and since the volume is asymptotically cubic, there must be some subsequence along which the area ratio $R^{-3} \cdot \text{Area}(\partial B(R))$ in (69) goes to zero. Thus with $\chi(M^4) = 1$, we obtain

$$\int |\text{Rm}|^2 dVol = 32\pi^2 + \frac{16\pi^2 k^2}{1-k^2} = 16\pi^2 \frac{2-k^2}{1-k^2} \quad (70)$$

□

Proposition 3.7 (Instanton Curvature Decay Rate) *In the generic case, that $k \in (-1, 0) \cup (0, 1)$, we have $|\text{Rm}| = O(R^{-2})$ where R is distance to the origin. When $k = 0$ we have $|\text{Rm}| = O(R^{-3})$, and when $k \pm 1$ we have $|\text{Rm}| = O(R^{-2})$ on all geodesics except along one a totally geodesic subvariety, that contains the origin, on which at least one sectional curvature does not decay.*

Proof. We computed that

$$|\text{Ric}|^2 dVol = \frac{8k^2 uv}{(1 + (1+k)u^2 + (1-k)v^2)^3} du \wedge dv \wedge d\theta^1 \wedge d\theta^2 \quad (71)$$

and

$$dVol = \frac{4}{M^2} (1 + (1+k)u^2 + (1-k)v^2) uv du \wedge dv \wedge d\theta^1 \wedge d\theta^2 \quad (72)$$

so that

$$|\text{Ric}| = \frac{\sqrt{2}kM}{(1 + (1+k)u^2 + (1-k)v^2)^2}. \quad (73)$$

Using the approximation (56) and the transitions $u = \sqrt[4]{\frac{\sqrt{2M}}{1+k}} \sqrt{\tilde{R}} \cos \psi$, $v = \sqrt[4]{\frac{\sqrt{2M}}{1-k}} \sqrt{\tilde{R}} \sin \psi$, we have $u^2 + v^2 = \tilde{R} \sqrt{\sqrt{2M}} \left(\frac{\cos^2 \psi}{\sqrt{1+k}} + \frac{\sin^2 \psi}{\sqrt{1-k}} \right)$. Thus we see that $|\text{Ric}| = O(\tilde{R}^{-2}) = O(R^{-2})$, except when $k = 0, 1, -1$. Therefore we have $|\text{Rm}| = O(R^{-2})$ except when $k = 0, 1, -1$.

When $k = 0$, Ric is zero, and therefore $K_\Sigma = O(R^{-3})$ gives $|\text{Rm}| = O(R^{-3})$.

When $k = \pm 1$, we have $|\text{Rm}| = O(R^{-2})$ generically except that $|\text{Rm}| = O(1)$ when $\eta = \pi/2$ (in the $k = 1$ case) or $\eta = 0$ (in the $k = -1$ case), where K_Σ does not decay. It is easy to see that this is a totally geodesic 2-dimensional submanifold, that it is the zero-set of a holomorphic vector field (one of the two \mathcal{X}_i) and is therefore a subvariety, and contains the origin. \square

3.4 Three kinds of Blowdown

Our investigation of the non-exceptional Taub-NUTs ends with an examination of their blowdown objects. The polytope itself can easily be shown to converge uniquely under blowdown, but the situation on the instanton itself is more complex. The distance function itself converges, but there is a complicating topological issue that stems from how the torus fibers converge.

3.4.1 The Gromov-Hausdorff Blowdown

The metric on the 4-manifold, given by (31), has four eigen-directions. The first two, of course, are $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial v}$, with eigenvalue $1 + (1+k)u^2 + (1-k)v^2$. The remaining eigenvalues are combinations of the fields $\mathcal{X}_1, \mathcal{X}_2$. The larger of the remaining eigenvectors is a combination of $\mathcal{X}_1, \mathcal{X}_2$ that varies from point to point, and is not a Killing field. The final, smallest eigenvector can be represented by an actual Killing field, the field $-(1-k)\frac{\partial}{\partial \theta^1} + (1+k)\frac{\partial}{\partial \theta^2}$.

This final eigendirection, a Killing field, is also the manifold's asymptotic “collapsing field” in precisely the sense of the F-structures of Cheeger-Gromov [4]. It grows only like $O(1)$ with distance, whereas the first three eigenvalues grow linearly with geometric distance. Thus when the metric is scaled down, the first three directions remain large, but the final direction collapses, suggesting that the 4-manifold collapsed to a 3-manifold. But that is only the case when the collapsing field, restricted to each torus fiber, is rational, meaning $\frac{1-k}{1+k} \in \mathbb{Q}$. When the field is not rational, the blowdown limit is 2-dimensional.

To see directly how this happens, consider the spheres of fixed radius 1 in M^4 , as the metric scales down. This is always a topological 3-sphere, but the metric is a Berger-like “squashed” metric as the Killing field $(1-k)\mathcal{X}_1 - (1+k)\mathcal{X}_2$, formerly having nearly unit-norm, has smaller and smaller norm, whereas the other two directions retain roughly unit length. In the case $k = 0$ the Killing field is precisely the Hopf field, and the limiting manifold is the round 2-sphere. When $\frac{1-k}{1+k}$ is rational, the limit is an orbifold which in this case is either a “football metric” with two singular points or a “raindrop metric” with one singular point, and which has

bounded but non-constant sectional curvature. When $\frac{1-k}{1+k}$ is irrational, the limit is a segment, and convergence remains convergence with bounded curvature.

With these considerations in hand, consider again the blowdown process of the larger 4-manifolds. We can see four distinct behaviors. First, in the case $k = 0$, which is the standard Ricci-flat Taub-NUT instanton, where sectional curvature decays like $O(R^{-3})$, the metric spheres collapse to round 2-spheres. The Gromov-Hausdorff convergence is to flat \mathbb{R}^3 .

Second, in the case $\frac{1-k}{1+k}$ is rational but non-zero, sectional curvature decays only like $O(R^{-2})$. Geodesic spheres converge to “footballs” or “raindrops,” so the manifold overall converges to a stratified orbifold. The orbifold may have two singular rays, each totally geodesic, that meet at the origin or just one singular ray (corresponding to the orbifold spheres being “footballs” or “raindrops”). The metric on the limiting 3-orbifold is smooth in the orbifold sense, everywhere except the origin. The origin has a curvature singularity, and nearby sectional curvature growth like $O(R^{-2})$.

Third, $\frac{1-k}{1+k}$ may be irrational but non-zero. The Gromov-Hausdorff convergence is to the closed quarter-plane $\{u \geq 0\} \cap \{v \geq 0\}$. Sectional curvature is bounded everywhere except at the origin, where it rises like $O(R^{-2})$.

The final case is $k = \pm 1$, which is explored in Section 4.

3.4.2 Two generalized blowdowns

It is both convenient and potentially enlightening to create a blow-down scheme that deals with all of the cases $k \in (-1, 1)$ in a single framework, and creates a 3-dimensional blowdown limit. To do so, first remove the $u = 0$ and $v = 0$ axes along with their fibers, so two 2-planes, intersecting at the origin, are removed from M^4 . Then pass to the universal cover. This has the effect of “unwrapping” the torus fibers, so we now have a non-complete 4-manifold with an isometric \mathbb{R}^2 action. The blow-down limit, in the Gromov-Hausdorff sense, is a 4-manifold, but is in fact the a topological product of a non-complete 3-manifold (that has one Killing direction) with a line—the line direction comes from the zero eigenvector of the G^{-1} portion of the metric.

We may throw away the line direction, and then complete the 3-manifold by compactifying the isometric direction by modding-out by translation by 2π along the flow of the remaining Killing field. The result will be a 3-dimensional conifold. There will be two totally-geodesic rays of conifold points with possibly irrational cone-angle, but with bounded sectional curvature. However at the origin will be a curvature singularity, and a generalized conifold point.

There is in fact a second generalized blowdown, which results in a 4-manifold. The result is flat \mathbb{R}^4 if and only if the original metric is the standard Ricci-flat Taub-NUT. The metric on the 4-fold is obtained by *not* throwing away the line direction as was done above, but after taking the Gromov-Hausdorff limit, we compactifying both directions. Rather than doing this, however, we do something equivalent but simpler. We take the Gromov-Hausdorff limit on the polytope, and where we reconstruct

the moment functions. Using the moment functions and the process of Section 2, we rebuild a 4-manifold.

3.4.3 Metric and coordinate convergence under blowdown

Consider the 4-manifold $(M^4, J, \omega, \mathcal{X}_1, \mathcal{X}_2)$ with its polytope metric given by $g_\Sigma = \frac{2\sqrt{2}}{M} (1 + (1+k)u^2 + (1-k)v^2) (du^2 + dv^2)$. Note that

$$g_4 = g_\Sigma + (G^{-1})_{ij} d\theta^i \otimes d\theta^j \quad (74)$$

where G^{-1} is the matrix from (31). Scaling the coordinates by setting $u = \sqrt[4]{\frac{M}{2\sqrt{2}}} \tilde{u}$, $v = \sqrt[4]{\frac{M}{2\sqrt{2}}} \tilde{v}$, the polytope metric becomes

$$g_\Sigma = \left(\sqrt{\frac{2\sqrt{2}}{M}} + (1+k)\tilde{u}^2 + (1-k)\tilde{v}^2 \right) (d\tilde{u}^2 + d\tilde{v}^2), \quad (75)$$

and we can send $M \rightarrow \infty$. The coordinates θ^1, θ^2 remain unscaled. Both metrics g_Σ and g_4 converge. We get $g_\Sigma = ((1+k)\tilde{u}^2 + (1-k)\tilde{v}^2) (d\tilde{u}^2 + d\tilde{v}^2)$, and G^{-1} becomes

$$G^{-1} = \frac{\frac{1}{4}\tilde{u}^2\tilde{v}^2(\tilde{u}^2 + \tilde{v}^2)}{(1+k)\tilde{u}^2 + (1-k)\tilde{v}^2} \begin{pmatrix} (1+k)^2 & 1-k^2 \\ 1-k^2 & (1-k)^2 \end{pmatrix}. \quad (76)$$

Note that $\det(G^{-1}) = 0$. Its zero eigenvector is $\vec{v} = (1-k)\frac{\partial}{\partial\theta^1} - (1+k)\frac{\partial}{\partial\theta^2}$ and its eigenvector of eigenvalue $\frac{\tilde{u}\tilde{v}((1+k)^2 + (1-k)^2)}{4((1+k)\tilde{u}^2 + (1-k)\tilde{v}^2)}$ is $\vec{v} = (1+k)\frac{\partial}{\partial\theta^1} + (1-k)\frac{\partial}{\partial\theta^2}$. Setting $\tilde{\theta} = \frac{1+k}{2}\theta^1 + \frac{1-k}{2}\theta^2$ gives a 3-dimensional metric of

$$g = ((1+k)\tilde{u}^2 + (1-k)\tilde{v}^2) (d\tilde{u}^2 + d\tilde{v}^2) + \frac{\tilde{u}^2\tilde{v}^2(\tilde{u}^2 + \tilde{v}^2)}{(1+k)\tilde{u}^2 + (1-k)\tilde{v}^2} d\tilde{\theta}^2 \quad (77)$$

which is the metric on the 3-conifold discussed above. We clearly still have a Killing field $\tilde{\mathcal{X}} = \partial/\partial\tilde{\theta}$, and we may take a Riemannian quotient to obtain a quarter-plane polytope. Its sectional curvature is

$$K_\Sigma = 2k \frac{(1+k)\tilde{u}^2 - (1-k)\tilde{v}^2}{((1+k)\tilde{u}^2 + (1-k)\tilde{v}^2)^3}. \quad (78)$$

It is also not difficult to compute the Ricci curvature of the conifold. It is diagonal in $(\tilde{u}, \tilde{v}, \tilde{\theta})$ coordinates, and is given by

$$\text{Ric}_3 = \begin{pmatrix} \frac{-4\sqrt{2}k}{(1+k)\tilde{u}^2 + (1-k)\tilde{v}^2} & 0 & 0 \\ 0 & \frac{4\sqrt{2}k}{(1+k)\tilde{u}^2 + (1-k)\tilde{v}^2} & 0 \\ 0 & 0 & \frac{2\sqrt{2}k(1+k)^2\tilde{u}^2\tilde{v}^2(\tilde{u}^2 - \tilde{v}^2)}{((1+k)\tilde{u}^2 + (1-k)\tilde{v}^2)^3} \end{pmatrix} \quad (79)$$

One notices immediately that scalar curvature is not zero.

Finally we consider what the possible moment functions may be on the limit. If we use (30) and simply plug in the scaled (u, v) coordinates, the resulting functions are multiples of each other. From (30) and plugging in \tilde{u}, \tilde{v} , then sending $M \rightarrow \infty$, we have $\varphi^1 = \frac{1+k}{2\sqrt{2}}\tilde{u}^2\tilde{v}^2$, $\varphi^2 = \frac{1-k}{2\sqrt{2}}\tilde{u}^2\tilde{v}^2$, and

we see the two rescaled moment functions are multiples of each other. We set $\tilde{\varphi}^1 = \frac{1}{2}\tilde{u}^2\tilde{v}^2$.

To obtain a second moment function, consider the function $\tilde{\varphi}^2 = -\frac{1}{2}(1-k)\varphi^1 + \frac{1}{2}(1+k)\varphi^2 = -\frac{1-k}{2M}v^2 + \frac{1+k}{2M}u^2$, which clearly coincides with the 0-eigenvector of the scaled metric. This functions does not scale appropriately, but artificially scaling it by \sqrt{M} gives $\tilde{\varphi}^2 = -\frac{1}{2}(1-k)\tilde{v}^2 + \frac{1}{2}(1+k)\tilde{u}^2$ in the limit. In this way we can build a 4-manifold.

Proposition 3.8 (Second Generalized Blowdown) *The blowdown polytope is the quarter-plane in (\tilde{u}, \tilde{v}) -coordinates. It has natural commuting momentum functions $\tilde{\varphi}^1 = \frac{1}{2}\tilde{u}^2\tilde{v}^2$, $\tilde{\varphi}^2 = -\frac{1}{2}(1+k)\tilde{u}^2 + \frac{1}{2}(1-k)\tilde{v}^2$. These correspond to the moment functions of a singular 4-manifold with a half-plane polytope, that has metric*

$$\begin{aligned} g_4 &= g_\Sigma + (G^{-1})_{ij}d\theta^i \otimes d\theta^j, \quad \text{where} \\ g_\Sigma &= ((1+k)\tilde{u}^2 + (1-k)\tilde{v}^2)(d\tilde{u} \otimes d\tilde{u} + d\tilde{v} \otimes d\tilde{v}) \\ G^{-1} &= \begin{pmatrix} \frac{\tilde{u}^2\tilde{v}^2(\tilde{u}^2+\tilde{v}^2)}{(1+k)\tilde{u}^2+(1-k)\tilde{v}^2} & \frac{-2k\tilde{u}^2\tilde{v}^2}{(1+k)\tilde{u}^2+(1-k)\tilde{v}^2} \\ \frac{-2k\tilde{u}^2\tilde{v}^2}{(1+k)\tilde{u}^2+(1-k)\tilde{v}^2} & \frac{(1+k)\tilde{u}^2+(1-k)\tilde{v}^2}{(1+k)\tilde{u}^2+(1-k)\tilde{v}^2} \end{pmatrix} \end{aligned} \quad (80)$$

Proof. Plugging in \tilde{u} and \tilde{v} we have transitions

$$\frac{1}{\sqrt[4]{\sqrt{2}M}}\tilde{u} = \sqrt{\sqrt{x^2+y^2}+y}, \quad \frac{1}{\sqrt[4]{\sqrt{2}M}}\tilde{v} = \sqrt{\sqrt{x^2+y^2}-y} \quad (81)$$

Thus we set $x = \frac{1}{\sqrt{\sqrt{2}M}}\tilde{x}$, $y = \frac{1}{\sqrt{\sqrt{2}M}}\tilde{y}$ to obtain identities

$$\begin{aligned} \tilde{u} &= \sqrt{\sqrt{\tilde{x}^2+\tilde{y}^2}+\tilde{y}}, \quad \tilde{v} = \sqrt{\sqrt{\tilde{x}^2+\tilde{y}^2}-\tilde{y}} \\ \frac{1}{2}(\tilde{u}^2+\tilde{v}^2) &= \sqrt{\tilde{x}^2+\tilde{y}^2}, \quad \frac{1}{2}(\tilde{u}^2-\tilde{v}^2) = \tilde{y}, \quad \tilde{u}\tilde{v} = \tilde{x}. \end{aligned} \quad (82)$$

Therefore

$$\begin{aligned} \tilde{\varphi}^1 &= \frac{1}{2}\tilde{u}^2\tilde{v}^2 = \frac{1}{2}\tilde{x}^2 \\ \tilde{\varphi}^2 &= -\frac{1}{2}(1-k)\tilde{v}^2 + \frac{1}{2}(1+k)\tilde{u}^2 = \tilde{y} + k\sqrt{\tilde{x}^2+\tilde{y}^2} \end{aligned} \quad (83)$$

This is obviously a map from the right half-plane in the (x, y) system to the right half-plane in the $(\tilde{\varphi}^1, \tilde{\varphi}^2)$ -system. We have transitions

$$A = \begin{pmatrix} \frac{\tilde{x}}{\sqrt{\tilde{x}^2+\tilde{y}^2}} & 0 \\ \frac{k\tilde{x}}{\sqrt{\tilde{x}^2+\tilde{y}^2}} & 1 + \frac{k\tilde{y}}{\sqrt{\tilde{x}^2+\tilde{y}^2}} \end{pmatrix} \quad (84)$$

Therefore the polytope metric, in (\tilde{x}, \tilde{y}) and in (\tilde{u}, \tilde{v}) coordinates is

$$\begin{aligned} g_\Sigma &= \frac{k\tilde{y} + \sqrt{\tilde{x}^2+\tilde{y}^2}}{\sqrt{\tilde{x}^2+\tilde{y}^2}}(d\tilde{x} \otimes d\tilde{x} + d\tilde{y} \otimes d\tilde{y}) \\ g_\Sigma &= ((1+k)\tilde{u}^2 + (1-k)\tilde{v}^2)(d\tilde{u} \otimes d\tilde{u} + d\tilde{v} \otimes d\tilde{v}). \end{aligned} \quad (85)$$

The corresponding 4-manifold metric is $g_4 = g_\Sigma + (G^{-1})_{ij}d\theta^i \otimes d\theta^j$. \square

3.4.4 Distance functions on the generalized blow-downs

Setting $S(\tilde{u}, \tilde{v}) = \frac{1}{2}\sqrt{1+k}\tilde{u}^2 + \frac{1}{2}\sqrt{1-k}\tilde{v}^2$ gives

$$dS = \sqrt{1+k}\tilde{u}d\tilde{u} + \sqrt{1-k}\tilde{v}d\tilde{v}, \text{ and } |dS|^2 = 1 \quad (86)$$

and therefore S is a distance function. Its zero locus is precisely $(\tilde{u}, \tilde{v}) = (0, 0)$, so S is the distance to the origin. Set $\tilde{u} = \frac{\sqrt{2}}{\sqrt[4]{1+k}}\sqrt{R}\cos\psi$, $\tilde{v} = \frac{\sqrt{2}}{\sqrt[4]{1-k}}\sqrt{R}\sin\psi$. Then $S(\tilde{u}, \tilde{v}) = R$.

We have

$$\nabla S = \frac{\sqrt{1+k}\tilde{u}}{(1+k)\tilde{u}^2 + (1-k)\tilde{v}^2} \frac{\partial}{\partial \tilde{u}} + \frac{\sqrt{1-k}\tilde{v}}{(1+k)\tilde{u}^2 + (1-k)\tilde{v}^2} \frac{\partial}{\partial \tilde{v}}. \quad (87)$$

This gives the system

$$\begin{aligned} \frac{d\tilde{u}}{dt} &= \frac{\sqrt{1+k}\tilde{u}}{(1+k)\tilde{u}^2 + (1-k)\tilde{v}^2} \\ \frac{d\tilde{v}}{dt} &= \frac{\sqrt{1-k}\tilde{v}}{(1+k)\tilde{u}^2 + (1-k)\tilde{v}^2} \end{aligned} \quad (88)$$

and we obtain unparametrized geodesics $\tilde{v} = C\tilde{u}^{\sqrt{\frac{1-k}{1+k}}}$. First pass parametrization:

$$\gamma(t) = (c_1 t^{\sqrt{1+k}}, c_2 t^{\sqrt{1-k}}) \quad (89)$$

We get $S(\gamma(t)) = \frac{1}{2}c_1^2\sqrt{1+k}t^{2\sqrt{1+k}} + \frac{1}{2}c_2^2\sqrt{1-k}t^{2\sqrt{1-k}}$

$$\frac{d}{dt} = c_1\sqrt{1+k}t^{\sqrt{1+k}-1} \frac{\partial}{\partial \tilde{u}} + c_2\sqrt{1-k}t^{\sqrt{1-k}-1} \frac{\partial}{\partial \tilde{v}} \quad (90)$$

4 The exceptional Taub-NUT

4.1 Coordinates

The exceptional Taub-NUT has moment functions, in terms of the volumetric normal coordinates

$$\varphi^1 = \frac{1}{\sqrt{2}}(-y + \sqrt{x^2 + y^2}) + \frac{\alpha}{2}x^2, \quad \varphi^2 = \frac{1}{\sqrt{2}}(y + \sqrt{x^2 + y^2}) \quad (91)$$

which is the case $k = 1$. In fact, simultaneous scaling in the (x, y) and (φ^1, φ^2) coordinates allows us to adjust α . We may make $\alpha = 2\sqrt{2}$, in which case (18) gives the polytope metric

$$g_\Sigma = \frac{1 + 2y + 2\sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2}} (dx \otimes dx + dy \otimes dy). \quad (92)$$

Making the transitions to (u, v) coordinates

$$u = \sqrt{2}\sqrt{\sqrt{x^2 + y^2} + y}, \quad v = \sqrt{2}\sqrt{\sqrt{x^2 + y^2} - y} \quad (93)$$

we obtain $\varphi^1 = \frac{1}{2\sqrt{2}}v^2(1+u^2)$ and $\varphi^2 = \frac{1}{2\sqrt{2}}u^2$. The metric transforms to

$$g_\Sigma = (1+u^2)(du \otimes du + dv \otimes dv). \quad (94)$$

The matrix $(G^{-1})_{ij} = \langle \mathcal{X}_i, \mathcal{X}_j \rangle$ is

$$G^{-1} = \begin{pmatrix} \frac{1}{2} \frac{v^2((1+u^2)^2 + u^2v^2)}{1+u^2} & \frac{1}{2} \frac{u^2v^2}{1+u^2} \\ \frac{1}{2} \frac{u^2v^2}{1+u^2} & \frac{1}{2} \frac{u^2}{1+u^2} \end{pmatrix} \quad (95)$$

so we have reconstructed the instanton metric: $g_4 = g_\Sigma + (G^{-1})_{ij}d\theta^i \otimes d\theta^j$. It is easy to compute the polytope sectional curvature in (u, v) coordinates: we have

$$K_\Sigma = -\frac{1-u^2}{(1+u^2)^3}. \quad (96)$$

Notice that $K_\Sigma = -1$ along the positive v axis. Thus the instanton (M^4, g_4) has $|\text{Rm}| = O(1)$ along all geodesic rays that map to this ray in the polytope; these rays in M^4 constitute a rotationally symmetric 2-dimensional submanifold that is totally geodesic (it is the zero-set of one of the Killing fields). Using (102) below, where we compute geodesics and the distance function R , this also implies that $K_\Sigma = O(R^{-2})$ along all other geodesics based at the origin.

4.2 Distance functions and geodesic normal coordinates

To find distance functions, we use the separation trick, as before. Setting $S(u, v) = f(u) + h(v)$ and finding

$$1 = |\nabla S|^2 = \frac{(f_u)^2 + (h_v)^2}{1+u^2} \quad (97)$$

we get

$$\begin{aligned} (f_u)^2 + (h_v)^2 &= (\cos^2(\eta) + u^2) + \sin^2(\eta) \\ f_u &= \sqrt{\cos^2(\eta) + u^2}, \quad h_v = \sin(\eta). \end{aligned} \quad (98)$$

So separating variables into $f_u = \sqrt{\cos^2(\eta) + u^2}$, $h_v = \sin(\eta)$ gives

$$\begin{aligned} S_\eta(u, v) &= \frac{\cos^2 \eta}{2} \left(\frac{u}{\cos \eta} \sqrt{1 + \frac{u^2}{\cos^2 \eta}} + \log \left(\frac{u}{\cos \eta} + \sqrt{1 + \frac{u^2}{\cos^2 \eta}} \right) \right) + v \sin(\eta) \end{aligned} \quad (99)$$

For each η , the characteristics of S_η provide one geodesic from the origin, found by solving $\frac{d\gamma}{dt} = \nabla S_\eta$ with $\gamma(0) = (0, 0)$. This gives the system

$$\frac{du}{dt} = \frac{\sqrt{\cos^2 \eta + u^2}}{1+u^2}, \quad \frac{dv}{dt} = \frac{\sin \eta}{1+u^2} \quad (100)$$

which is already partially separated. Rather than integrate the first equation, we eliminate the t parameter to obtain the unparametrized geodesic equation

$$\frac{dv}{du} = \frac{\sin \eta}{\sqrt{\cos^2 \eta + u^2}}, \text{ or } v = \sin(\eta) \log \left(\frac{u}{\cos \eta} + \sqrt{1 + \frac{u^2}{\cos^2 \eta}} \right). \quad (101)$$

Now integrating the first equation in (100) we get

$$\begin{aligned} R &= \frac{1}{2}u\sqrt{\cos^2 \eta + u^2} + \frac{2 - \cos^2 \eta}{2} \log \left(\frac{u}{\cos \eta} + \sqrt{1 + \frac{u^2}{\cos^2 \eta}} \right) \\ &= \frac{1}{2}u\sqrt{\cos^2 \eta + u^2} + \frac{v}{2} \frac{1 + \sin^2 \eta}{\sin \eta} \end{aligned} \quad (102)$$

and so we have recovered the distance function $R = R(u, v)$, which is evaluated explicitly by solving for η from (101), then plugging into (102).

Let \tilde{R}, ψ be “almost polar coordinates” where $\tilde{R} = \frac{1}{2}u^2 + v$ and $u = \sqrt{2} \cos(\psi) \sqrt{\tilde{R}}$, $v = \sin^2(\psi) \tilde{R}$. We show that this “almost distance function” \tilde{R} approximates the distance function R to within a factor of 2.

Proposition 4.1 (Almost distance function) *Let $R = R(u, v)$ be the distance function and let $\tilde{R} = \frac{1}{2}u^2 + v$ be the almost distance function. Then for sufficiently large \tilde{R} (say $\tilde{R} > 100$) we have*

$$\frac{R}{\tilde{R}} \in [1, 2]. \quad (103)$$

Proof. Put $u = \sqrt{2} \cos(\psi) \tilde{R}^{1/2}$, $v = \sin^2(\psi) \tilde{R}$. Using (102) and (101) the following two equations determine the transition between the (R, η) and the (\tilde{R}, ψ) systems:

$$\begin{aligned} R &= \tilde{R} \left[\cos \psi \sqrt{\frac{\cos^2 \eta}{2\tilde{R}} + \cos^2 \psi} + \frac{\sin^2 \psi}{\sin \eta} \frac{1 + \sin^2 \psi}{2} \right], \\ \frac{\sin^2 \psi}{\sin \eta} &= \tilde{R}^{-1} \left[\log \sqrt{\tilde{R}} + \log \left(\frac{\sqrt{2} \cos \psi}{\cos \eta} + \sqrt{\frac{1}{\tilde{R}} + \frac{2 \cos^2 \psi}{\cos^2 \eta}} \right) \right]. \end{aligned} \quad (104)$$

From $\frac{1}{\sin \eta} \geq 1$, $\frac{1 + \sin^2 \psi}{2} \geq 1$, and $\frac{\cos^2 \eta}{2\tilde{R}} + \cos^2 \psi \geq \cos^2 \psi$, we get

$$\begin{aligned} \frac{R}{\tilde{R}} &= \cos \psi \sqrt{\frac{\cos^2 \eta}{2\tilde{R}} + \cos^2 \psi} + \frac{\sin^2 \psi}{\sin \eta} \frac{1 + \sin^2 \psi}{2} \\ &\geq \cos^2 \psi + \sin^2 \psi = 1. \end{aligned} \quad (105)$$

The opposite inequality is slightly more involved, as it is not possible to uniformly bound the ratio $\cos \psi / \cos \eta$. We perform the estimate in two parts: if $\cos \eta \leq \frac{1}{\sqrt{2}}$ then $\sin \eta \geq \frac{1}{\sqrt{2}}$ and we have simply

$$\begin{aligned} \frac{R}{\tilde{R}} &= \cos \psi \sqrt{\frac{\cos^2 \eta}{2\tilde{R}} + \cos^2 \psi} + \frac{\sin^2 \psi}{\sin \eta} \frac{1 + \sin^2 \psi}{2} \\ &\leq (1 + \epsilon(\tilde{R})) \cos^2 \psi + \sin^2 \psi \left(\frac{1 + \sin^2 \psi}{\sqrt{2}} \right) \leq \sqrt{2} \end{aligned} \quad (106)$$

If $\cos \eta \geq \frac{1}{\sqrt{2}}$ then

$$\begin{aligned} \frac{\sin^2 \psi}{\sin \eta} &= \tilde{R}^{-1} \left[\log \sqrt{\tilde{R}} + \log \left(\frac{\sqrt{2} \cos \psi}{\cos \eta} + \sqrt{\frac{1}{\tilde{R}} + \frac{2 \cos^2 \psi}{\cos^2 \eta}} \right) \right] \\ &\leq \tilde{R}^{-1} \left[\log \sqrt{\tilde{R}} + \log \left(2 \cos \psi + \sqrt{\frac{1}{\tilde{R}} + 4 \cos^2 \psi} \right) \right] \\ &\leq \epsilon(\tilde{R}) + (1 + \epsilon(\tilde{R})) \log(4 \cos \psi) \leq \log 5 \end{aligned} \quad (107)$$

so that

$$\begin{aligned} \frac{R}{\tilde{R}} &= \cos \psi \sqrt{\frac{\cos^2 \eta}{2\tilde{R}} + \cos^2 \psi} + \frac{\sin^2 \psi}{\sin \eta} \frac{1 + \sin^2 \psi}{2} \\ &\leq (1 + \epsilon(\tilde{R})) \cos^2 \psi + \log(5) \leq 1 + \log(5). \end{aligned} \quad (108)$$

□

4.3 Volume and curvature computations

The “almost ball” of radius \tilde{R} , denoted $AB(\tilde{R})$, is the set of points with $u, v \geq 0$ and $v + \frac{1}{2}u^2 \leq \tilde{R}$. Using $\det G^{-1} = \frac{1}{4}u^2v^2$, we obtain the 4-manifold volume form $dVol = \frac{1}{2}uv(1 + u^2)du \wedge dv \wedge d\theta^1 \wedge d\theta^2$. The almost-ball’s volume is therefore

$$Vol AB(\tilde{R}) = 4\pi^2 \int_0^{\sqrt{2\tilde{R}}} \int_0^{\tilde{R} - \frac{1}{2}u^2} \frac{1}{2}uv(1 + u^2) dv du = \frac{\pi^2}{6} (R^4 + 2R^3). \quad (109)$$

Proposition 4.2 *The exceptional Taub-NUT isstanton has quartic asymptotic volume growth: $Vol B(R) = O(R^4)$.*

Proof. Combine (109) with Proposition 4.1. □

Next we compute the Ricci potentials and the Ricci pseudo-volume form. Using that $\sqrt{\mathcal{V}} = \sqrt{\text{Det } g^{-1}} = \frac{1}{2}uv$, from (19) we have

$$\begin{aligned} \mathcal{R}_1 &= -\langle \nabla \log \mathcal{V}, \nabla \varphi^1 \rangle = \frac{1}{\sqrt{2}} \frac{1 + u^2 + v^2}{1 + u^2} \\ \mathcal{R}_2 &= -\langle \nabla \log \mathcal{V}, \nabla \varphi^2 \rangle = \frac{1}{\sqrt{2}} \frac{1}{1 + u^2}. \end{aligned} \quad (110)$$

Taking exterior derivatives and using (20), we have

$$\begin{aligned} |\text{Ric}|^2 dVol &= \frac{2uv}{(1 + u^2)^3} du \wedge dv \wedge d\theta^1 \wedge d\theta^2, \\ |\text{Ric}|^2 &= \frac{4}{(1 + u^2)^4}. \end{aligned} \quad (111)$$

Integrating over the (u, v) quarter-plane clearly gives an infinite value. Alternatively one can integrate over the pseudo-ball of radius R to determine $\int_{B(R)} |\text{Ric}|^2 dVol = O(R^2)$.

Notice that (111) gives that $|\text{Ric}| = O(1)$ along $u = 0$. Also notice, using (102), that along all other geodesics we have $|\text{Ric}| = O(R^{-2})$.

4.4 A scaled and an unscaled pointed limit

4.4.1 The blowdown

Scaling the matrix by $\frac{1}{M^4}$ and scaling u and v by M , leaving θ^1 unscaled and scaling θ^2 by M^{-2} , and sending $M \rightarrow \infty$ we obtain the polytope metric

$$g_\Sigma = u^2 (du^2 + dv^2) \quad (112)$$

and the matrix $(G^{-1})_{ij} = \langle \mathcal{X}_i, \mathcal{X}_j \rangle$ becomes

$$G^{-1} = \frac{1}{2} \begin{pmatrix} v^2(u^2 + v^2) & v^2 \\ v^2 & 1 \end{pmatrix}. \quad (113)$$

The resulting instanton metric $g_4 = g_\Sigma + (G^{-1})_{ij} d\theta^i \otimes d\theta^j$ is singular along the axis $u = 0$. The blowdown has a 2-dimensional submanifold along which we have both a topological and a curvature singularity. The polytope sectional curvature is $K_\Sigma = -u^{-4}$, so the v -axis clearly holds singular curvature values.

4.4.2 An unscaled pointed limit

The instanton (M^4, g_4) has rays along which sectional curvatures equal to -1 . A natural question is what happens when we take a pointed limit along such a ray. We shall see that the resulting limit is the exceptional half-plane instanton.

The sectional curvature is constant along the v -axis, so we rechoose coordinates to center ourselves farther and farther along. For any $A > 0$ set

$$\tilde{u} = u, \quad \tilde{v} = v - A. \quad (114)$$

The range of these coordinates is $\tilde{u} \in [0, \infty]$ and $\tilde{v} \in [-A, \infty]$ so in the limit the range is the entire half-plane. In the Gromov-Hausdorff limit, the torus fibers actually become cylinders: the \mathcal{X}_2 direction converges on circumference $1/2$ while the \mathcal{X}_1 direction becomes infinite. The field \mathcal{X}_1 itself becomes infinitely long, so we make an affine recombination to rechoose it. For each A , choose new Killing fields

$$\tilde{\mathcal{X}}_1 = \frac{1}{2A} (\mathcal{X}_1 - 2\sqrt{2}A^2 \mathcal{X}_2), \quad \tilde{\mathcal{X}}_2 = \sqrt{2}\mathcal{X}_2. \quad (115)$$

The polytope metric converges to

$$g_\Sigma = (1 + \tilde{u}^2) (d\tilde{u}^2 + d\tilde{v}^2) \quad (116)$$

and one can check directly that the matrix G^{-1} converges to

$$G^{-1} = \frac{1}{1 + \tilde{u}^2} \begin{pmatrix} (1 + \tilde{u}^2)^2 + 4\tilde{u}^2\tilde{v}^2 & 2\tilde{u}^2\tilde{v} \\ 2\tilde{u}^2\tilde{v} & \tilde{u}^2 \end{pmatrix} \quad (117)$$

(we omit the tedious but straightforward computation). Finally, notice that for the new Killing fields in (115) we have new moment functions $\tilde{\varphi}^1, \tilde{\varphi}^2$ defined up to a constant. Choosing the constant appropriately, the transitions from old to new moment functions are $\tilde{\varphi}^1 = \frac{1}{2A}(\varphi^1 - A^2(1 + 2\sqrt{2}\varphi^2))$, $\tilde{\varphi}^2 = \sqrt{2}\varphi^2$. These functions also converge, and in the limit we obtain

$$\tilde{\varphi}^1 = \tilde{v} + \tilde{v}\tilde{u}^2, \quad \tilde{\varphi}^2 = \frac{1}{2}\tilde{u}^2. \quad (118)$$

Comparing this data to the data laid out in Section 5 we see that this is indeed the same metric (with the momentum variables switched). However the half-plane instanton is not simply connected—its fibers are tori—whereas the limit we have taken *is* simply connected—its fibers are cylinders. Thus this limit is actually the universal cover of the exceptional half-plane instanton described below.

5 The exceptional half-plane instanton

The exceptional half-plane instanton is given by

$$\varphi^1 = \frac{1}{2}x^2, \quad \varphi^2 = y + yx^2. \quad (119)$$

Using (18) we obtain the polytope metric

$$g_\Sigma = (1 + x^2)(dx \otimes dx + dy \otimes dy) \quad (120)$$

and the matrix $G^{-1} = \langle \mathcal{X}_i, \mathcal{X}_j \rangle$ is

$$G^{-1} = \frac{1}{1 + x^2} \begin{pmatrix} x^2 & 2x^2y \\ 2x^2y & (1 + x^2)^2 + 4x^2y^2 \end{pmatrix} \quad (121)$$

One notices the formal similarity with the exceptional Taub-NUT instanton. There are two differences: the domains of the variables, and the size of the torus fibers. Notice that G^{-1} for the half-plane and the exceptional Taub-NUT are substantively different.

The formal similarity between this metric and the exceptional Taub-NUT metric allows us to use all of the polytope formalism, except that the domain is now a half-plane instead of a quarter-plane. On the instanton itself, the origin is not fixed by the torus action, but moves in a circle.

We have Ricci potentials $\mathcal{R}_1 = \frac{2}{1+x^2}$ and $\mathcal{R}_2 = \frac{4y}{1+x^2}$. Then from (20) the norm-square of Ricci curvature is

$$\begin{aligned} |\text{Ric}|^2 dVol &= \frac{8x}{(1+x^2)^3} dx \wedge dy \wedge d\theta^1 \wedge d\theta^2 \\ |\text{Ric}|^2 &= \frac{8}{(1+x^2)^4} \end{aligned} \quad (122)$$

Once again we have that $|\text{Ric}| = 1$ along geodesics within the 2-dimensional submanifold given by $x = 0$ (which is itself totally a geodesic submanifold), and we have $|\text{Ric}| = O(R^{-2})$ along all other geodesics.

A final question of interest is whether or not the exceptional half-plane instanton (more precisely its universal cover) actually *is* the exceptional Taub-NUT. They have different polytopes, but conceivably these bear a relationship to each other like the flat half-plane polytope and the flat quarter-plane polytope for $\mathbb{C} \times \mathbb{C}$. That is, perhaps they are the same 4-manifold, except one has a two translational fields whereas the other has one rotational and one translational field.

That this is *not* the case can be seen by considering the exceptional 2-submanifold in each. In both cases, this 2-submanifold has a sectional curvature that does not decay. But this sectional curvature is an *orthogonal* sectional curvature. The exceptional 2-submanifold in the case of the exceptional half-plane instanton itself has a flat metric, whereas the exceptional 2-submanifold in the exceptional Taub-NUT is not zero, but decays like $O(R^{-2})$ at infinity.

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